SYNTACTIC GENERIC CONSTRUCTIONS AND EHRENFEUCHT THEORIES

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Semantic generic constructions

Let $\mathbf{K}_0$ be a class of finite structures of a countable predicate language. The class $\mathbf{K}_0$ is endowed with a partial order relation $\leqslant$ which is invariant under the transition to isomorphic structures, connoting the property of being a *self-sufficient structure*, or *strong substructure*, and satisfying the following axioms:

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(1) if $\mathcal{A} \leqslant \mathcal{B}$, then $\mathcal{A} \subseteq \mathcal{B}$;

(2) if $\mathcal{A} \leqslant \mathcal{C}$, $\mathcal{B} \in \mathbf{K}_0$, and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{A} \leqslant \mathcal{B}$;

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(3) $\varnothing$ is the least element of the system $(\mathbf{K}_0; \leqslant)$;

(4) (the *amalgamation property*) for any structures $\mathcal{A}, \mathcal{B}$, $\mathcal{C} \in \mathbf{K}_0$, having embeddings $f_0 : \mathcal{A} \to \mathcal{B}$ and $g_0 : \mathcal{A} \to \mathcal{C}$ such that $f_0(\mathcal{A}) \leqslant \mathcal{B}$ and $g_0(\mathcal{A}) \leqslant \mathcal{C}$, there are a structure $\mathcal{D} \in \mathbf{K}_0$ and embeddings $f_1 : \mathcal{B} \to \mathcal{D}$ and $g_1 : \mathcal{C} \to \mathcal{D}$ for which $f_1(\mathcal{B}) \leqslant \mathcal{D}$, $g_1(\mathcal{C}) \leqslant \mathcal{D}$ and $f_0 \circ f_1 = g_0 \circ g_1$.

With the class $\mathbf{K}_0$ determined from finite structures of $\mathbf{K}_0$ using *amalgamation* (i. e., embedding the structures $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$ in structures $\mathcal{D}$ so as to comply with the amalgamation property), we construct a countable $(\mathbf{K}_0; \leqslant)$-*generic model* $\mathcal{M}$ step by step so as to satisfy the following:

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- for any finite substructure $\mathcal{A}\subseteq\mathcal{M}$, there is a structure $\mathcal{B}\in\mathbf{K}_0$, $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{M}$, for which $\mathcal{B}\leqslant\mathcal{M}$, i. e., $\mathcal{B}\leqslant\mathcal{B}'$ for any structure $\mathcal{B}'\in\mathbf{K}_0$ with $\mathcal{B}\subseteq\mathcal{B}'\subseteq\mathcal{M}$;

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- for any finite substructure $\mathcal{A} \subseteq \mathcal{M}$ and any structure $\mathcal{B} \in \mathbf{K}_0$ such that $\mathcal{A} \leqslant \mathcal{B}$, there is a structure $\mathcal{B}' \leqslant \mathcal{M}$ for which $\mathcal{B} \simeq_{\mathcal{A}} \mathcal{B}'$.

THEOREM (J.Baldwin, N.Shi)

For any partially ordered class $(\mathbf{K}_0; \leqslant)$, *satisfying conditions* 1–4, *there exists a* $(\mathbf{K}_0; \leqslant)$*-generic model.*

Historical review

E. Hrushovski, using a modification of generic *Jonsson — Fraïssé construction*, has disproved *Zil'ber Conjecture* constructing examples of strongly minimal not locally modular theories in which infinite groups are not interpreted. His original construction, which served as a basis for building of appropriate examples and solving other known model-theoretic problems, has given an impetus to intensive studies of both the *Hrushovski construction* together with its various (in a broad sense) modifications, capable of creating "pathological" theories with given properties and axiomatic bases, allowing to determine applicability bounds for that construction.

Historical review

- R.Fraïssé (France), general principles, Fraïssé limit;
- B.Jonsson (Sweden), homogeneous-universal models;
- E.Hrushovski (Israel), strongly minimal theories, geometries, fusions of fields, superstable ω-categorical theories;
- J.T.Baldwin (USA), projective planes, general principles, properties, classifications, fields, geometries, abstract elementary classes;
- B.Poizat (France), general principles, properties, classifications, geometries, fields;
- D.W.Kueker (USA), C.Laskowski (USA), general principles, properties;
- F.Wagner (France), general principles, properties;
- B.Herwig (Germany), weight ω in small stable theories;
- M.Itai (USA), projective planes;
- N.Shi (USA), general principles, properties;

Historical review

- B.Zilber (Great Britain), geometries, fields;
- A.Pillay (USA), simple theories, preservation of ω-categoricity;
- A.Tsuboi (Japan), preservation of ω-categoricity, strong amalgamation property;
- M.Ziegler (Germany), fields, fusions;
- A.Baudisch (Germany), groups, fields, fusions;
- Z.Chatzidakis (France), simple theories;
- S.Shelah (Israel), abstract elementary classes;
- M.J.de Bonis (USA), A.Nesin (Turkey), almost strongly minimal generalized n-gons;
- K.Holland (USA), fields, fusions, model completeness;
- V.V.Verbovskiy (Kazakhstan), elimination of imaginaries, CM-triviality;

Historical review

- K.Ikeda (Japan), projective planes, strong amalgamation property;
- H.Kikyo (Japan), strong amalgamation property;
- I.Yoneda (Japan), CM-triviality;
- M.Pourmahdian (Iran), simple theories;
- D.M.Evans (Great Britain), ω-categorical structures;
- A.Hasson (Great Britain), interpretations of structures with the definable multiplicity property, fusions;
- A.M.Vershik (Russia), isometries;
- S.Solecki (USA), isometries;
- A.Martin-Pizarro (Germany), fields, fusions;
- M.Hils (Germany), fusions.

We fix an at most countable language L and consider a class $\mathbf{T}_0$ of (complete or incomplete) types $\Phi(A)$ (without free variables) over finite sets A such that $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$ for any quantifier-free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$. Suppose that the class $\mathbf{T}_0$ is equipped with a partial order $\leqslant$, closed under bijective substitutions $[\Phi(A)]^{A}_{A'}$ of pairwise distinct constants in A' for constants in A into types $\Phi(A) \in \mathbf{T}_0$. Furthermore, we assume that results of bijective substitutions $[\Phi(A)]^{A}_{X}$ of sets X of variables for constants in A into types $\Phi(A) \in \mathbf{T}_0$ (over all sets A) form a countable set.

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- if $\Phi \leqslant \mathrm{X}$, $\Psi \in \mathbf{T}_0$, and $\Phi \subseteq \Psi \subseteq \mathrm{X}$, then $\Phi \leqslant \Psi$;
- some type $\Phi_0(\varnothing)$ is the least element of the system $(\mathbf{T}_0; \leqslant)$;
- (the *t-amalgamation property*) for any types $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C) \in \mathbf{T}_0$, if there exist injections $f_0 : A \to B$ and $g_0 : A \to C$ with $[\Phi(A)]^A_{f_0(A)} \leqslant \Psi(B)$ and $[\Phi(A)]^A_{g_0(A)} \leqslant \mathrm{X}(C)$, then there are a type $\Theta(D) \in \mathbf{T}_0$ and injections $f_1 : B \to D$ and $g_1 : C \to D$ for which $[\Psi(B)]^B_{f_1(B)} \leqslant \Theta(D)$, $[\mathrm{X}(C)]^C_{g_1(C)} \leqslant \Theta(D)$ and $f_0 \circ f_1 = g_0 \circ g_1$;

- (the *local realizability property*) if $\Phi(A) \in \mathbf{T}_0$ and $\Phi(A) \vdash \exists x\, \varphi(x)$ (respectively, t is a term of language $L \cup A$ containing no free variables), then there are a type $\Psi(B) \in \mathbf{T}_0$, $\Phi(A) \leqslant \Psi(B)$, and an element $b \in B$ for which $\Psi(B) \vdash \varphi(b)$ $((t \approx b) \in \Psi(B))$;

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- (the *t-uniqueness property*) for any types $\Phi(A), \Psi(A) \in \mathbf{T}_0$ if the set $\Phi(A) \cup \Psi(A)$ is consistent then $\Phi(A) = \Psi(A)$.

Self-sufficiency

Let $\mathbf{T}_0$ be a class of types, $\mathbf{P}$ be a class of models, and $\mathcal{M}$ be a model in $\mathbf{P}$. The class $\mathbf{T}_0$ is *cofinal* in the model $\mathcal{M}$ if, for each finite set $A \subseteq M$, there are a finite set B, $A \subseteq B \subseteq M$, and a type $\Phi(B) \in \mathbf{T}_0$ such that $\mathcal{M} \models \Phi(B)$. The class $\mathbf{T}_0$ is *cofinal* in $\mathbf{P}$ if $\mathbf{T}_0$ is cofinal in every model of $\mathbf{P}$. We denote by $\overline{\mathbf{T}}_0$ the class of all models $\mathcal{M}$ with the condition that $\mathbf{T}_0$ is cofinal in $\mathcal{M}$, and by $\mathbf{P}$ a subclass of $\overline{\mathbf{T}}_0$ such that each type $\Phi \in \mathbf{T}_0$ is true for some model in $\mathbf{P}$.

Self-sufficiency

Let $\mathcal{M}$ be a model in $\overline{\mathbf{T}}_0$, A and B be finite sets in $\mathcal{M}$ with $A \subseteq B$. We call A a *strong subset* of the set B (in the model $\mathcal{M}$), and write $A \leqslant B$, if there exist types $\Phi(A), \Psi(B) \in \mathbf{T}_0$, for which $\Phi(A) \leqslant \Psi(B)$ and $\mathcal{M} \models \Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0 \subseteq M$ (in the model $\mathcal{M}$), where $A \subseteq M_0$, if $A \leqslant B$ for any finite set B such that $A \subseteq B \subseteq M_0$ and $\Phi(A) \subseteq \Psi(B)$ for some types $\Phi(A), \Psi(B) \in \mathbf{T}_0$ with $\mathcal{M} \models \Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A \leqslant M_0$.

If $A \leqslant M$ in $\mathcal{M}$ then we refer to A as a *self-sufficient set* (in $\mathcal{M}$).

Generic models

A model $\mathcal{M} \in \mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{T}_0; \leqslant)$ if any finite set $A \subseteq M$ is contained in some self-sufficient set in $\mathcal{M}$. A class $\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{T}_0; \leqslant)$ if each model in $\mathbf{P}$ has finite closures.

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A countable model $\mathcal{M} \in \overline{\mathbf{T}}_0$ is $(\mathbf{T}_0; \leqslant)$-*generic* if it satisfies the following conditions:
(a) $\mathcal{M}$ has finite closures;
(b) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{T}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leqslant \Psi(B)$, then there exists a set $B' \leqslant M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

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For any $(\mathbf{K}_0; \leqslant)$*-generic model* $\mathcal{M}$, *there exists a quantifier-free class* $(\mathbf{T}_0; \leqslant')$ *such that* $\mathcal{M}$ *is* $(\mathbf{T}_0; \leqslant')$*-generic.*

Self-sufficient classes

A generic class $(\mathbf{T}_0; \leqslant)$ is *self-sufficient* if the following axiom holds:

- if $\Phi, \Psi, \mathrm{X} \in \mathbf{T}_0$, $\Phi \leqslant \Psi$, and $\mathrm{X} \subseteq \Psi$, then $\Phi \cap \mathrm{X} \leqslant \mathrm{X}$.

t-Covering property

Below we denote by $(\mathbf{T}_0; \leqslant)$ a self-sufficient generic class, by $\overline{\mathcal{M}}$ a $(\mathbf{T}_0; \leqslant)$-generic model, by T a theory $\mathrm{Th}(\overline{\mathcal{M}})$, and by $\mathbf{K}$ a subclass of $\overline{\mathbf{T}}_0$ consisting of all models of the theory T.
A self-sufficient class $(\mathbf{T}_0; \leqslant)$ has the *t-covering property* if

- each type $\Phi(X)$ of theory T is deduced from some type $[\Psi_\Phi(B)]^B_{X\cup Y}$, where $\Psi_\Phi(B) \in \mathbf{T}_0$.

Closures

Let $\mathbf{K}$ be a class having finite closures, $\mathcal{M}$ be a model in $\mathbf{K}$, and S be a set in $\mathcal{M}$. The least (by inclusion) closed set in $\mathcal{M}$, containing S, is called an *intrinsic closure* of S in $\mathcal{M}$ and is denoted by $\mathrm{icl}_{\mathcal{M}}(S)$, or by $\overline{S}$, if it is clear from the context which of the models $\mathcal{M}$ is in point. If the set $\overline{S}$ is finite then it is referred to as a *self-sufficient closure* of the set S. A type in the class $\mathbf{T}_0$, corresponding to the self-sufficient closure $\overline{A}$ of a set A, is denoted by $\overline{\Phi}(\overline{A})$. If $\Phi(A) \in \mathbf{T}_0$ and $\mathcal{M} \models \Phi(A)$, then the type $\overline{\Phi}(\overline{A})$ is called a *self-sufficient closure* of the type $\Phi(A)$.

THEOREM

If the class $\mathbf{K}$ *has finite closures then for any model* $\mathcal{M} \in \mathbf{K}$ *and any finite set* $A \subseteq M$ *there exists a self-sufficient closure* $\overline{A}$ *of* A*. Moreover,* $\overline{A} \subseteq \mathrm{acl}_{\mathcal{M}}(A)$*.*

Closures and homogeneity

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COROLLARY.

If the class **K** *has finite closures then the generic model* $\overline{\mathcal{M}}$ *is homogeneous.*

Genericity of countable homogeneous models

A generic class $(\mathbf{T}_0; \leqslant)$ is *hereditary* if $\mathbf{T}_0$ consists of types $\Phi(A)$ containing all possible formulas describing a number of copies of a system of elements of a set B over a system of elements of a set A, and interrelations of elements of copies for each set $B \supseteq A$, where a respective type $\Psi(B)$ belongs to $\mathbf{T}_0$ and satisfies $\Phi(A) \leqslant \Psi(B)$.

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Every at most countable, homogeneous (saturated) algebraic system $\mathcal{M}$ is a $(\mathbf{T}_0; \leqslant)$-generic model for some hereditary generic class $(\mathbf{T}_0; \leqslant)$ (with the t-covering property).

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COROLLARY.

Every complete countable theory is generic.

The uniform t-amalgamation property and saturated generic models

Let $(\mathbf{T}_0; \leqslant)$ be a self-sufficient class satisfying the following conditions:

- for any type $\Phi(A) \in \mathbf{T}_0$, the type $\overline{\Phi}(\overline{A})$ yields a formula $\chi_{\overline{\Phi}}(\overline{A})$ describing the self-sufficient condition for the closure $\overline{\Phi}(\overline{A})$; moreover, $\chi_{\overline{\Phi}}(\overline{A})$ also contains a formula which is deducible from $\Phi(A)$ and describes an upper bound for the cardinality of the set $\overline{A}$;

The uniform t-amalgamation property and saturated generic models

- for any self-sufficient types $\overline{\Phi}(\overline{A})$ and $\overline{\Psi}(\overline{B})$, where $\overline{\Phi}(\overline{A}) \leqslant \overline{\Psi}(\overline{B})$, and for any formula $\psi(X, Y)$ in $\overline{\Psi}(X \cup Y)$ (here, X and Y are disjoint sets of variables, bijective with sets $\overline{A}$ and $\overline{B} \setminus \overline{A}$ respectively), there exists a formula $\varphi(X)$ which is deducible from $\overline{\Phi}(X)$ and is such that the following formula holds true in $\overline{\mathcal{M}}$:

$$\forall X\, ((\chi_{\overline{\Phi}}(X) \wedge \varphi(X)) \to \exists Y\, (\chi_{\overline{\Psi}}(X, Y) \wedge \psi(X, Y))).$$

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$$\forall X\, ((\chi_{\overline{\Phi}}(X) \wedge \varphi(X)) \to \exists Y\, (\chi_{\overline{\Psi}}(X, Y) \wedge \psi(X, Y))).$$

If the above conditions are satisfied then we say that the class $(\mathbf{T}_0; \leqslant)$ has the *uniform t-amalgamation property*.

The uniform t-amalgamation property and saturated generic models

THEOREM

If $(\mathbf{T}_0; \leqslant)$ *is a self-sufficient class having the uniform* t*-amalgamation property and the class* $\mathbf{K}$ *has finite closures, then the* $(\mathbf{T}_0; \leqslant)$*-generic model* $\overline{\mathcal{M}}$ *is* ω*-saturated. Moreover, any finite set* $A \subseteq \overline{M}$ *is extendable to its self-sufficient closure* $\overline{A} \subseteq \overline{M}$*, the type* $\mathrm{tp}(\overline{A})$ *contains the type* $\overline{\Phi}(Y)$ *for a self-sufficient type* $\overline{\Phi}(\overline{A})$*, and* $\overline{\Phi}(Y) \vdash \mathrm{tp}(\overline{A})$*.*

Fusions

Let $(\mathbf{T}_0; \leqslant_0)$, $(\mathbf{T}_1; \leqslant_1)$, and $(\mathbf{T}_2; \leqslant_2)$ be generic classes of languages L_0, L_1, and L_2 respectively, $L_0 = L_1 \cap L_2$, $\leqslant_0 = \leqslant_1 \cap \leqslant_2$. A generic class $(\mathbf{T}_3; \leqslant_3)$ of language $L_1 \cup L_2$, such that $(\mathbf{T}_3; \leqslant_3) \upharpoonright L_i = (\mathbf{T}_i; \leqslant_i)$, $i = 1, 2$, is said to be a *fusion* of classes $(\mathbf{T}_1; \leqslant_1)$ and $(\mathbf{T}_2; \leqslant_2)$ over $(\mathbf{T}_0; \leqslant_0)$. In this case, a $(\mathbf{T}_3; \leqslant_3)$-generic model is a *fusion* of $(\mathbf{T}_1; \leqslant_1)$- and $(\mathbf{T}_2; \leqslant_2)$-generic models.

Hrushovski style fusions

Hrushovski style fusions of generic classes are defined by non-negative linear prerank functions δ_i for classes $(\mathbf{T}_i; \leqslant_i)$, $i = 0, 1, 2$, with non-negative linear prerank functions

$$\delta(A) = \delta_1(A) + \delta_2(A) - \delta_0(A),$$

of fusions, where $\delta_i(A) = |A| - \alpha_i \cdot |R_i(A)|$, $\alpha_i \in \mathbb{R}^+$, $R_i(A)$ is the number of tuples, being connected by predicates on A, $i = 0, 1, 2$.

Theories with finitely many countable models

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Remind the characterization for $I(T,\omega)=1$ (T is a countably categorical theory).

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If $1<I(T,\omega)<\omega$ then the theory T is called *Ehrenfeucht*.

PROBLEM

ON CHARACTERIZATION OF EHRENFEUCHT THEORIES.

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LACHLAN PROBLEM

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A theory is called *stable* if it doesn't have formulas $\varphi(\bar{x}, \bar{y})$ and tuples $\bar{a}_n, \bar{b}_n$, $n \in \omega$, such that

$$\models \varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j.$$

Historical review

- A.Ehrenfeucht (Poland), 1961 (examples);
- R.Vaught (USA), 1961 ($I(T,\omega) \neq 2$);
- M.Morley (USA), 1965, J.T.Baldwin (USA), A.H.Lachlan (Canada), 1971 ($I(T,\omega) = 1$ or $\geq \omega$ for uncountably categorical theories);
- E.A.Palyutin (USSR), 1971 (countably categorical universals);
- A.H.Lachlan (Canada), 1973 ($I(T,\omega) = 1$ or $\geq \omega$ for superstable theories);
- M.G.Peretyat'kin (USSR), 1973 (decidable Ehrenfeucht theories, new examples), 1980 (constant expansions and Ehrenfeuchtness);
- M.Benda (Czechoslovakia), 1974 (Ehrenfeuchtness implies existence of powerful types);
- D.Lascar (France), 1976 ($I(T,\omega) = 1$ or $\geq \omega$ for superstable theories), 1982 (finite Rudin—Keisler preorders for Ehrenfeucht theories);

Historical review

- R.Woodrow (Canada), 1976 (sufficient conditions for theories to be like Ehrenfeucht example with three countable models), 1978, (constant expansions and Ehrenfeuchtness);
- S.Shelah (Israel), 1978 ($I(T,\omega)=1$ or $\geq\omega$ for superstable theories);
- A.Pillay (Great Britain, USA), 1978 ($I(T,\omega)\geq 4$ for theories with infinite constantly defined sets), 1980 (dense partial order for Ehrenfeucht theories with small number of links), 1983 ($I(T,\omega)=1$ or $\geq\omega$ for normal theories), 1989 ($I(T,\omega)=1$ or $\geq\omega$ for 1-based theories),
- T.G.Mustafin (USSR), 1981 ($I(T,\omega)=1$ or $\geq\omega$ for theories with superstable types);
- J.Saffe (Germany), 1981 ($I(T,\omega)=1$ or $\geq\omega$ for superstable theories);
- T.Millar (USA), 1981 (constant expansions and Ehrenfeuchtness); 1985 (decidable Ehrenfeucht theories);

Historical review

- B.Omarov (USSR), 1983 (constant expansions and Ehrenfeuchtness);
- C.J.Ash, T. Millar (USA), 1983 (constructive models of Ehrenfeucht theories);
- A.Tsuboi (Japan), 1985 (any Ehrenfeucht theory being a union of ω-categorical theories is unstable), 1986 ($I(T,\omega)=1$ or $\geq\omega$ for unions of pseudo-superstable theories);
- S.Thomas (USA), 1986 (constant expansions and Ehrenfeuchtness);
- E.Hrushovski (Israel), 1989 ($I(T,\omega)=1$ or $\geq\omega$ for finitely based theories);
- A.A.Vikent'ev (USSR), 1989 (inheritance of non-Ehrenfeuchtness from non-Ehrenfeucht formula restrictions);

Historical review

- R.Reed (USA), 1991 (decidable Ehrenfeucht theories);
- B.Herwig (Germany), J.Loveys (USA), A.Pillay (USA), P.Tanović (Yugoslavia), F.Wagner (Germany), 1992 ($I(T,\omega)=1$ or $\geq\omega$ for stable theories without dense forking chains);
- S.S.Goncharov (Russia), M.Pourmahdian (Iran), 1995 (finiteness of rank for any Ehrenfeucht theory);
- B.Herwig (Germany), 1995 (small stable theories with infinite weight);
- B.Khoussainov, A.Nies (New Zealand), R.A.Shore (USA), 1997 (recursive models of Ehrenfeucht theories);
- K.Ikeda (Japan), A.Pillay (USA), A.Tsuboi (Japan), 1998 (dense linear orders in almost ω-categorical theories with three countable models);

Historical review

- B.Kim (USA), 1999 ($I(T,\omega)=1$ or $\geq\omega$ for supersimple theories);
- P.Tanović (Yugoslavia), 2001 ($I(T,\omega)\geq\omega$ for stable theories with an infinite set of pairwise different constants);
- S.Lempp, T.Slaman, 2004 (Π^1_1-completeness of Ehrenfeucht property);
- W.Calvert, V.S.Harizanov, J.F.Knight, S.Miller (USA), 2005 (the complexity of index sets of classical Ehrenfeucht theories);
- P.Tanović (Serbia), 2006 (a countable, complete, first-order theory with infinite $\mathrm{dcl}(\varnothing)$ and precisely three non-isomorphic countable models interprets a variant of Ehrenfeucht's or Peretyat'kin's example);

Historical review

- P.Tanović (Serbia), 2009 (a presence of types directed by constants guaranties the maximal number of non-isomorphic countable models of theory;
  proof of the PILLAY CONJECTURE: if T is the elementary diagram of a countable model then $I(T, \omega) \geq \omega$);
- A.N.Gavryushkin (Russia), 2006–2009 (computable models of Ehrenfeucht theories).

Ehrenfeucht example

$\mathcal{M} = \langle \mathbb{Q}, <, c_n \rangle_{n\in\omega}$, $c_n < c_{n+1}$,

$I(\mathrm{Th}(\mathcal{M}), \omega) = 3$:

$\lim\limits_{n\to\infty} c_n = \infty$ (the prime model);

$\lim\limits_{n\to\infty} c_n \in \mathbb{Q}$ (the prime model over a realization of nonisolated 1-type);

$\lim\limits_{n\to\infty} c_n \in \mathbf{Ir}$ (the saturated model).

A type $p(\bar{x}) \in S(T)$ is called *powerful* type of theory T if for any models $\mathcal{M}$ of T realizing p the model $\mathcal{M}$ realizes any type $q \in S(T) : \mathcal{M} \models S(T)$.
If $I(T, \omega) < \omega$ then T has a powerful type.

Properties of theories with finitely many countable models

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If $I(T, \omega) < \omega$ then T has a powerful type.

An existence of powerful type implies the *smallness* of theory T i.e. the set $S(T)$ is countable. It also implies that there are prime models $\mathcal{M}_{\bar{a}}$ over tuples $\bar{a}$ for any type $p \in S(T)$ and any its realization $\bar{a}$. Since all prime models over realizations of p are isomorphic, these models are denoted by $\mathcal{M}_p$.

Basic characteristics for theories with finitely many countable models

For any types $p, q \in S(T)$ we write $p \leq_{RK} q$ and say that p *is not more than* q *under the Rudin — Keisler preorder* if $\mathcal{M}_q$ has a realization of type p. At the same time we write $\mathcal{M}_p \leq_{RK} \mathcal{M}_q$ if $p \leq_{RK} q$. By $\mathrm{RK}(T)$ we denote the set of all isomorphism types of models $\mathcal{M}_p$ with the RK-relation induced by the relation $\leq_{RK}$ for models $\mathcal{M}_p$.

We say that models $\mathcal{M}_p$ and $\mathcal{M}_q$ are *RK-equivalent* if

$$\mathcal{M}_p \leq_{RK} \mathcal{M}_q \text{ and } \mathcal{M}_q \leq_{RK} \mathcal{M}_p.$$

Isomorphism types $\mathbf{M}_1$ and $\mathbf{M}_2$ from $\mathrm{RK}(T)$ are *RK-equivalent*:

$$\mathbf{M}_1 \sim_{RK} \mathbf{M}_2,$$

if their representatives are RK-equivalent.

Basic characteristics for theories with finitely many countable models

A model $\mathcal{M}$ is (*strongly*) *limit over a type p* if $\mathcal{M}$ is a union of an elementary chain $(\mathcal{M}_n)_{n\in\omega}$ such that $\mathcal{M}_n \simeq \mathcal{M}_p$, $n \in \omega$, and $\mathcal{M} \not\simeq \mathcal{M}_p$.
Let $\mathrm{RK}(T)$ be a finite system. For any class $\widetilde{\mathbf{M}} \in \mathrm{RK}(T)/{\sim_{RK}}$ consisting of isomorphism types of RK-equivalent models $\mathcal{M}_{p_1}, \ldots, \mathcal{M}_{p_n}$ we denote by $\mathrm{IL}(\widetilde{\mathbf{M}})$ the number of pairwise non-isomorphic limit models each of which is limit over some type p_i.

Syntactic characterization of theories with finitely many countable models

THEOREM

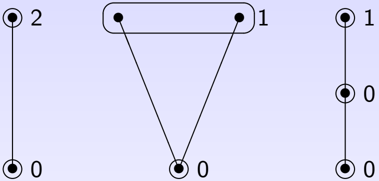
For any countable complete theory T the following conditions are equivalent:
(1) $I(T,\omega) < \omega$;
(2) T *is small,* $|\mathrm{RK}(T)| < \omega$ *and* $\mathrm{IL}(\widetilde{\mathbf{M}}) < \omega$ *for any* $\widetilde{\mathbf{M}} \in \mathrm{RK}(T)/\sim_{RK}$.
If the condition (1) (*or* (2)) *is true, then* T *satisfies the following conditions:*
(a) $\mathrm{RK}(T)$ *has the least element* $\mathbf{M}_0$ (*the isomorphism type of a prime model*) *and* $\mathrm{IL}(\widetilde{\mathbf{M}_0}) = 0$;
(b) $\mathrm{RK}(T)$ *has the greatest* $\sim_{RK}$*-class* $\widetilde{\mathbf{M}_1}$ (*the class of isomorphism types of all prime models over realizations of powerful types*), *and* $|\mathrm{RK}(T)| > 1$ *implies* $\mathrm{IL}(\widetilde{\mathbf{M}_1}) \geq 1$;
(c) *if* $|\widetilde{\mathbf{M}}| > 1$ *then* $\mathrm{IL}(\widetilde{\mathbf{M}}) \geq 1$.
Moreover the following decomposition formula is true:

$$I(T,\omega) = |\mathrm{RK}(T)| + \sum_{i=0}^{|\mathrm{RK}(T)/\sim_{RK}|-1} \mathrm{IL}(\widetilde{\mathbf{M}_i}),$$

where $\widetilde{\mathbf{M}_0}, \ldots, \widetilde{\mathbf{M}_{|\mathrm{RK}(T)/\sim_{RK}|-1}}$ *are all elements of the partially ordered set* $\mathrm{RK}(T)/\sim_{RK}$.

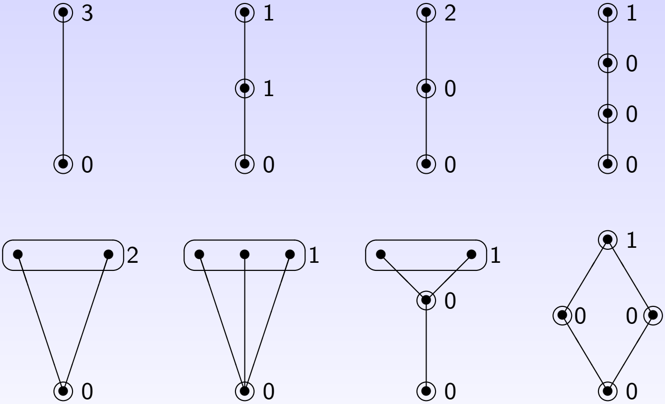
Examples

$I(T,\omega) = 3$



$I(T,\omega) = 4$

Examples



$$I(T,\omega)=5$$

Realization of basic characteristics for theories with finitely many countable models

THEOREM

Let $\langle X;\leq\rangle$ be a finite preordered set with the least element x_0 and the greatest class $\widetilde{x_1}$ in the ordered factor-set $\langle X;\leq\rangle/\sim$ by the relation $\sim$ (where $x\sim y \Leftrightarrow x\leq y$ and $y\leq x$), $f: X/\sim\,\to\omega$ be a function (a distribution function) satisfying the following conditions:
(a) $f(\widetilde{x_0})=0$;
(b) $|X|>1$ *implies* $f(\widetilde{x_1})\geq 1$.
(c) $|\widetilde{y}|>1$ *implies* $f(\widetilde{y})\geq 1$.
Then there exists a stable (unstable) theory T and an isomorphism $g:\langle X;\leq\rangle\,\widetilde{\to}\,\mathrm{RK}(T)$ such that $\mathrm{IL}(g(\widetilde{y}))=f(\widetilde{y})$ for any $\widetilde{y}\in X/\sim$.

Solution of the Lachlan Problem

THEOREM

For any $n \in \omega \setminus \{0, 2\}$ *there exists a stable theory* T_n *with* $I(T_n, \omega) = n$.

Language

- unary disjoint predicates Col_m, $m \in \omega$, and disjoint $P_1, \ldots, P_n$, $\vdash \forall x \bigvee\limits_{i=1}^{n} P_i(x)$, with given number n of prime models over realizations of non-principal 1-types $p_1(x), \ldots, p_n(x)$;

Language

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- the countable set of pairwise disjoint antisymmetric irreflexive binary relations Q_n, $n \in \omega$ defining acyclic digraphs with unbounded lengths of shortest Q^*-routes ($Q^* \rightleftharpoons \bigcup\limits_{n \in \omega} Q_n$) on the structures of $p_i(x)$ and on their neighbourhoods;

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- the countable set of pairwise disjoint symmetric irreflexive binary relations $P_{i,k,l}$, $i, k \in \omega$, $l = 1, \ldots, n$, allowing to connect elements a of infinite color $\left(\models \bigwedge\limits_{m\in\omega} \neg\mathrm{Col}_m(a)\right)$ with elements of finite colors m by principal formulas over a;

Language

- the countable set of pairwise disjoint symmetric irreflexive binary relations R_j, $j \in \omega$, connecting only elements of the same color and the same P_i, guaranteing the coincidence of prime models over realizations of $p_i(x)$ if these realizations are connected by R_j;

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- predicates R'_s, $s \in \omega$, guaranteing realization-equivalence of $\bigvee\limits_{i=1}^{n} p_i$ with all nonprincipal types.

Tools and objects

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- corealization amalgams.

Book

The results and generalizations for the class of all small theories are presented in:
[*Sudoplatov S.V.* The Lachlan Problem. — Novosibirsk, 2008. — 246 p.]

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The book is available in:
http://www.math.nsc.ru/∼sudoplatov/lachlan_03_09_2008.pdf
(in Russian),
http://www.math.nsc.ru/∼sudoplatov/lachlan_eng_03_09_2008.pdf
(in English).