Reduced symmetric presentations

A finite presentation $\mathcal{P} = \langle X; R \rangle$ is *symmetric* if every $r \in R$ is cyclically reduced and for every such r the set R contains r^{-1} and all cyclic permutations of r.

$\mathcal{P}$ is *reduced* if $x \neq 1$ for every $x \in X$ and no relators from R contain proper subwords trivial in $G_{\mathcal{P}} = \langle X; R \rangle$.

Notice, that every finitely presented group $G_{\mathcal{P}}$ has a reduced finite presentation $\mathcal{P}'$. Moreover, if the Word Problem is decidable in $G_{\mathcal{P}}$ then one can find $\mathcal{P}'$ effectively.

We always assume that presentations are finite, symmetric, and reduced.

Random $\mathcal{P}$-trivial words

Main idea: *Represent words trivial in $G_{\mathcal{P}}$ by Van Kampen diagrams*

Lemma (van Kampen). Let $\mathcal{P} = \langle X; R \rangle$ and $w \in (X^{\pm 1})^*$. Then $w =_{G_{\mathcal{P}}} 1$ if and only if there exists a diagram D over $\mathcal{P}$ with w written on the boundary of D.

Now the question is: what are random Van Kampen diagrams and what is a measure (asymptotic density) on them?

Random generators of Van Kampen diagrams

Main idea: Use *random generators* or *evolutionary models* to describe *random* van Kampen diagrams.

A diagram random generator RG builds up all diagrams over $\langle X; R \rangle$ (up to isomorphism) with positive probability starting with a single vertex and randomly (with prescribed probabilities) adding cells and edges, and randomly folding some of the edges with the same label.

Random generators as Markov's chains

The generator RG can be described as a simple random walk W on the (infinite) transition graph T of the random generator RG:

Vertices of T are Van Kampen diagrams over $\langle X; R\rangle$.

Two vertices v_1 and v_2 are connected by a directed edge (v_1, v_2) iff v_2 can be obtained from v_1 by either adding an edge or a cell, or performing a fold.

Theorem. [Completeness of RG] The graph T is connected.

Corollary. Algorithm RG produces any Van Kampen diagram over $\langle X; R\rangle$ with non-zero probability.

Measure on diagrams

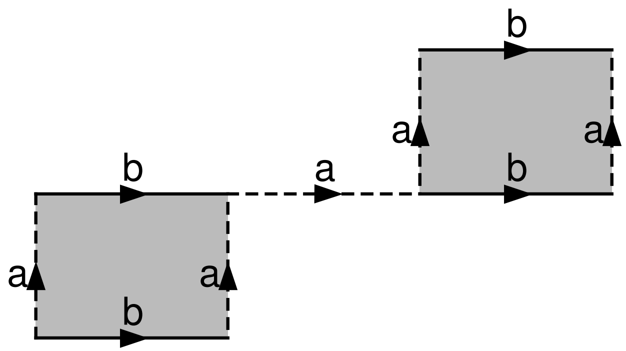
The standard Kolmogorov's measure on the set of all trajectories of T induces a measure λ on the set of vertices of T.

Define $\mu(D)$ of a diagram D as the sum of the probabilities assigned to the vertices in T which are isomorphic to D.

μ is a measure on the set Ω of all isomorphism classes of Van Kampen diagrams over $\langle X; R\rangle$.

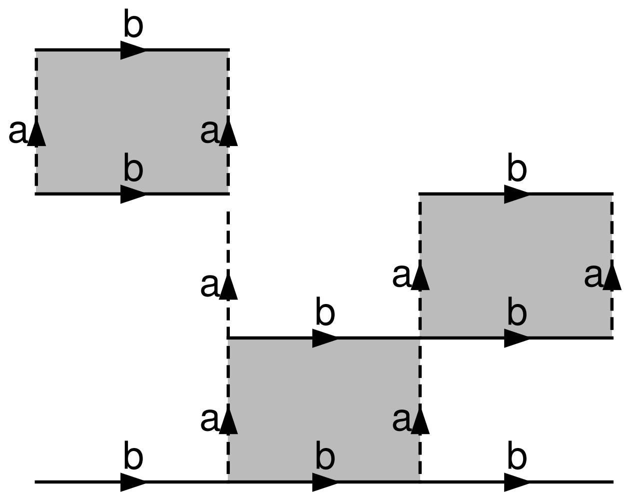
Geometric size of van Kampen diagrams.

$Area(D)$ of a diagram D is the total number of cells and free edges (i.e., edges which are not on boundaries of cells) in D.



$\chi(D_1) = 3$

in $A_2 = < a,b\ ;\ a\ b\ a^{-1}\ b^{-1} >$



$\chi(D_2) = 6$

in $A_2 = < a,b\ ;\ a\ b\ a^{-1}\ b^{-1} >$

Partition of diagrams.

The set Ω of all diagrams over $\langle X; R \rangle$ can be partitioned as

$$\Omega = \cup_{i=0}^{\infty} \Omega_i,$$

where Ω_n is the set of all diagrams D with $Area(D) = n$.

We use the asymptotic density on Ω with respect to the measure μ and the partition above.

Generic area of van Kampen diagrams.

Theorem[Myasnikov, Ushakov] *The set of all van Kampen diagrams* D *over* $\langle X; R\rangle$ *such that* $Area(D) \leq 4Length(\partial D)$ *is strongly generic in* Ω.

Corollary. *The Dehn function is linear on a strongly generic subset of* Ω.

Depth of diagrams

Let D be van Kampen diagram.

A sequence of vertices $v_1, \ldots, v_q$ in D is called a *cell-vertex chain* if for any $i = 1, \ldots, q-1$ there is a cell or a free edge c such that $v_i, v_{i+1} \in \partial c$.

Let c be an edge or a cell in D. The depth $\delta(c)$ of c in D is the length of the shortest cell-vertex chain from c to ∂D.

The depth of D is

$$Depth(D) = \max\{\delta(c, \partial D) \mid c \in C(D) \cup FE(D)\}$$

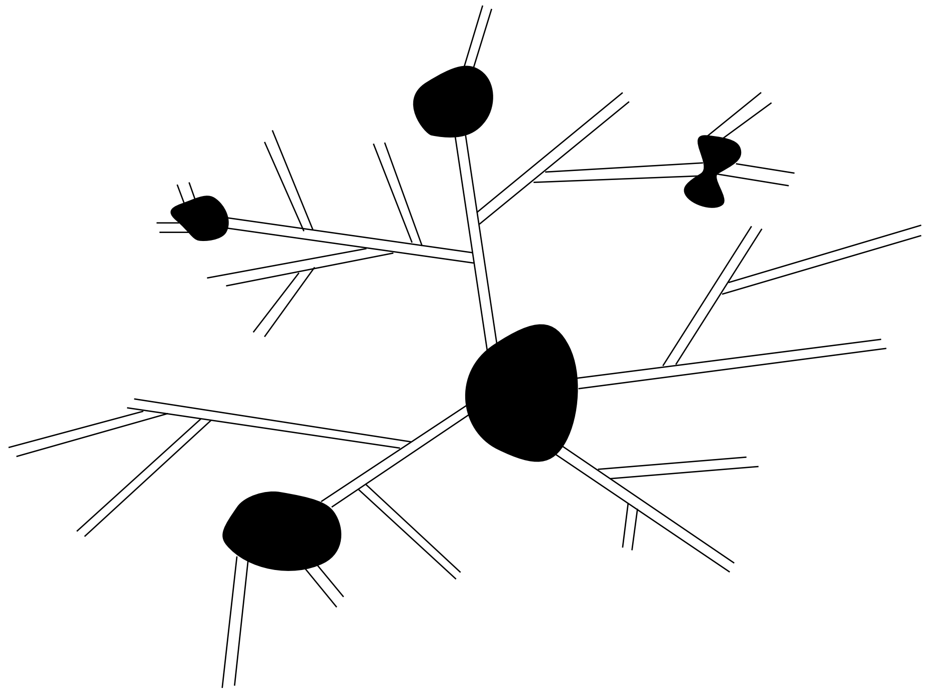
.

Generic depth of van Kampen diagrams.

Theorem [Myasnikov, Ushakov] *The set of all van Kampen diagrams D over $\langle X; R\rangle$ with $Depth(D) \leq 2 \log Area(D)$ is generic in Ω.*

Corollary *The set of all van Kampen diagrams D over $\langle X; R\rangle$ with $Depth(D) \leq 2 \log 4l(\partial D)$ is generic in Ω.*

Structure of a random van Kampen diagram.



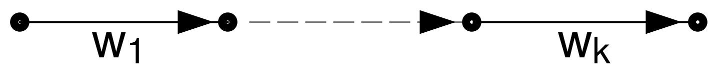
A new Search Algorithm for WP in groups.

Let $G = \langle X; R \rangle$.

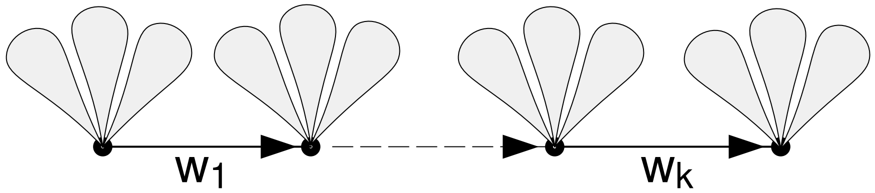
We describe a deterministic algorithm $\mathcal{A}_W$ which for a given finite symmetric presentation $\langle X; R \rangle$ and a word w in the alphabet $X^{\pm 1}$ halts and returns the answer Yes if and only if w represents the trivial element in the group $G = \langle X; R \rangle$.

In this event the algorithm $\mathcal{A}_W$ also gives a decomposition of w as a product of conjugates of relators from R.

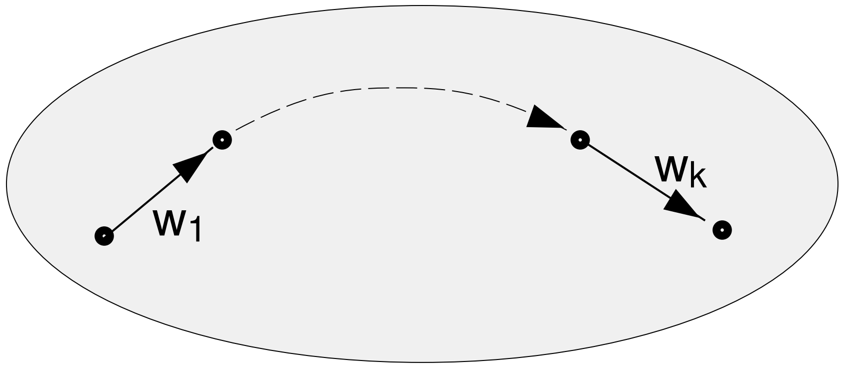
Moreover, there is a generic subset T of WP_{yes} and a polynomial $p(n)$ such that for a word $w \in T$ the algorithm $\mathcal{A}_W$ stops and gives the answer in at most $p(|w|)$ steps, where $|w|$ is the length of the word w.



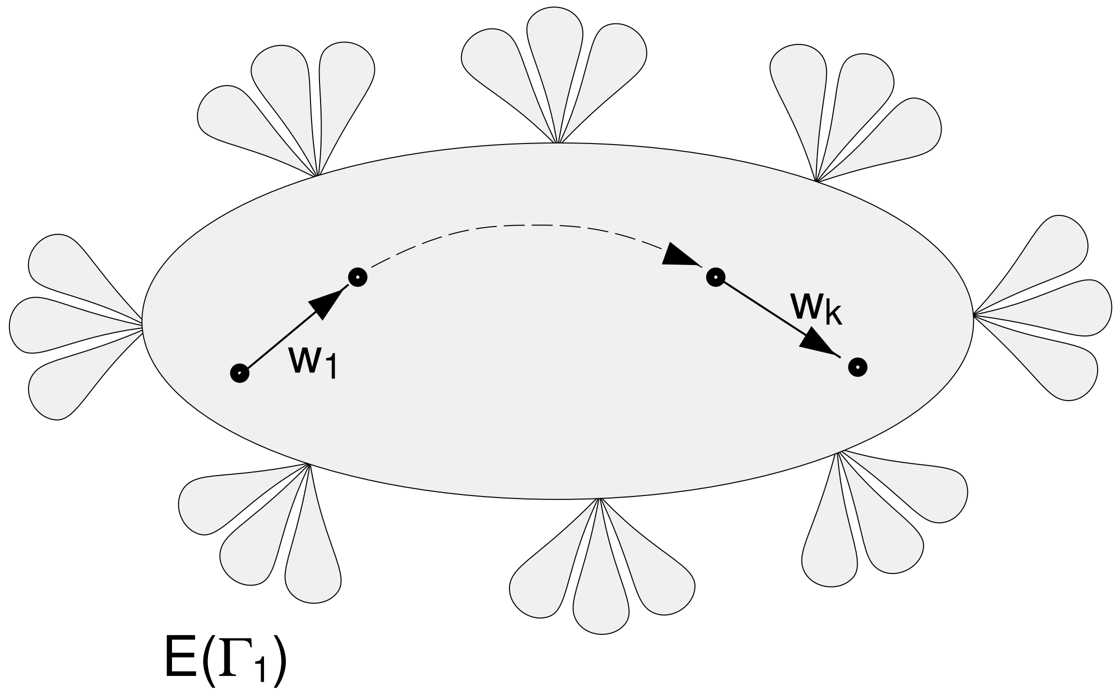
Γ_0



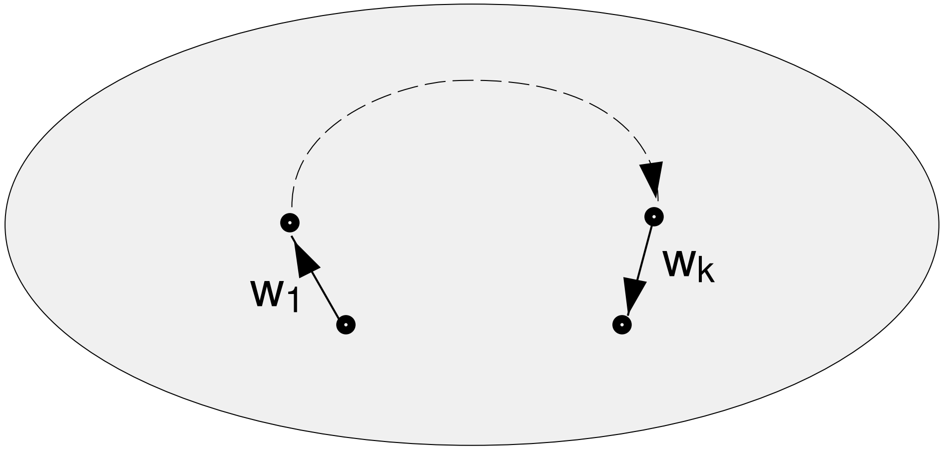
$E(\Gamma_0)$



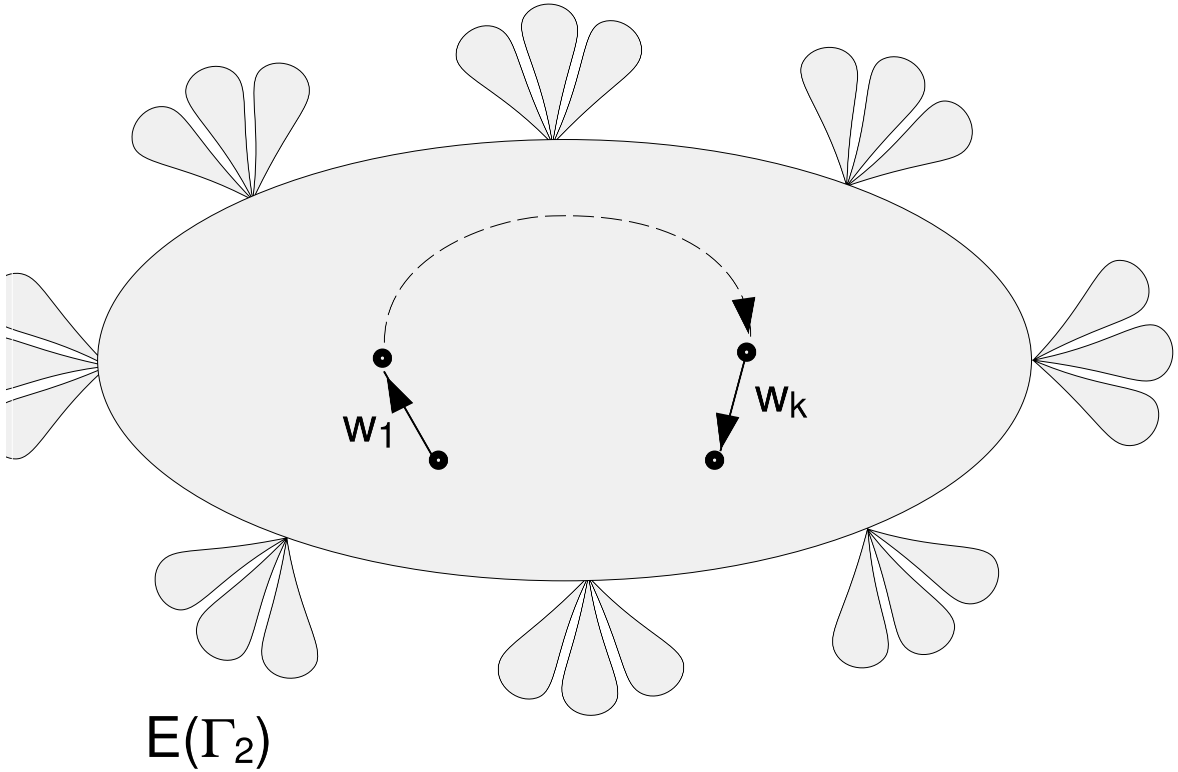
$$\Gamma_1 = S(E(\Gamma_0))$$

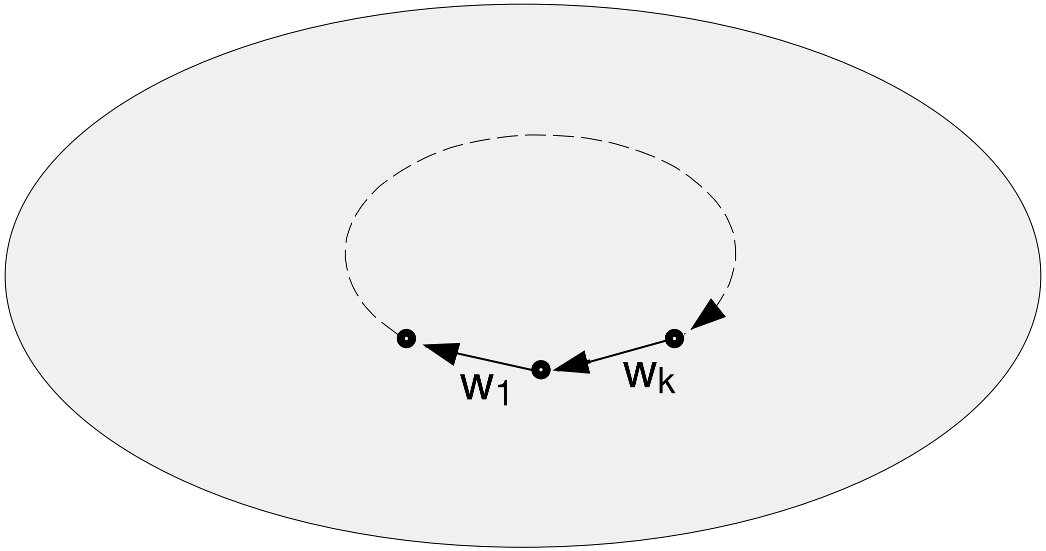


w_1
w_k
$E(\Gamma_1)$



$\Gamma_2 = S(E(\Gamma_1))$


w1
wk
E(Γ2)



$\Gamma_3 = S(E(\Gamma_2))$

The time complexity of $\mathcal{A}_W$.

Algorithm $\mathcal{A}_W$: Inductively compute

$$\Gamma_i = S(E(\Gamma_{i-1}))$$

until the original path w becomes a loop.

Output: the resulting graph Γ_i and a representation of w as a product of conjugates of relators from R.

Theorem. [Myasnikov, Ushakov] The algorithm $\mathcal{A}_W$ solves the search word problem for G in

$$O(|w| L(R)^{Depth(w)})$$

steps on an input $w =_G 1$, where $L(R)$ is the total length of the presentation of G.

The generic time complexity of $\mathcal{A}_W$.

Theorem. [Myasnikov, Ushakov] *The algorithm $\mathcal{A}_W$ is generically polynomial on WP_{yes} for every finite reduced presentation $G = \langle X; R \rangle$.*

The result follows from the exponential estimate on the worst-case time complexity of $\mathcal{A}_W$ and the genericity of the set of diagrams with logarithmic depth.

Basic extension B_S

Let $\langle X; R\rangle$ be a finite presentation. Denote by $\mathcal{L}$ a set of representatives (up to isomorphism) of all diagrams D over $\langle X; R\rangle$.

The basic extension B_S depends on a set of parameters S (the probabilities with which it adds cells or edges to a diagram and makes foldings) which allow one to obtain random diagrams with different properties.

Marked diagrams

Let D be a diagram with two distinguished subsets of vertices:

$M(D)$ - "marked vertices" (worked out vertices);

$A(D) \subseteq \partial D - M(D)$ - "active vertices" (vertices in the working).
Below always $|A(D)| \leq 1$.

Let $S = (s_1, s_2, s_3, s_4)$ be a sequence of reals such that $s_i \in [0,1], s_1 + s_2 + s_3 = 1$.

Basic Extension B_S:
Input: Let D be an extended van Kampen diagram over $\langle X; R \rangle$ such that either $A(D) \neq \emptyset$ or $\partial D - M(D) \neq \emptyset$.
Output: Diagram $B_S(D) = D$.

1) If $|A(D)| = 1$ then take the only vertex $v \in A(D)$. If $|A(D)| = 0$ then, choose randomly and uniformly an unmarked vertex $v \in \partial D - M(D)$ and put $A(D) = \{v\}$.

2) If v is not the last unmarked vertex in ∂D then with probability s_1 do a), with probability s_2 do b), and with probability $s_3 = 1 - s_1 - s_2$ do c) below:

a) Take randomly and uniformly a relator $r \in R$. Make a face N with the boundary label r at some vertex $u \in \partial N$. Attach N to v by identifying v with u. Go to 5).

b) Generate randomly and uniformly a letter $y \in X^{\pm 1}$. Make a free edge $e = (u_1, u_2)$ with the label y. Attach e to v by identifying v with u_1. Go to 5).

c) Do not attach anything to v and go to 4).

3) If v is the last unmarked vertex in D and $s_1 + s_2 \neq 0$ then with probability $\frac{s_1}{s_1+s_2}$ do a) below, otherwise do b):

   a) Take randomly and uniformly a relator $r \in R$. Make a face N with the boundary label r at some vertex $u \in \partial N$. Attach N to v by identifying v with u. Go to 5).

   b) Generate randomly and uniformly a letter $y \in X^{\pm 1}$. Make a free edge $e = (u_1, u_2)$ with the label y. Attach e to v by identifying v with u_1. Go to 5).

4) a) Let $(e_1, h_1), \ldots, (e_k, h_k)$ be all pairs of edges incident to v and such that for each i the following conditions hold:

- the path $e_i h_i$ belongs to the boundary of the diagram (with respect to a fixed orientation);

- all endpoints of e_i and f_i are unmarked;

- e_i and h_i^{-1} have the same labels (potential fold);

- edges e_i and h_i are not free.

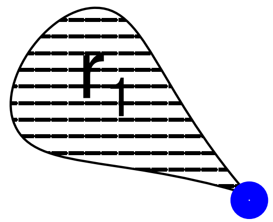
Then for each $i = 1, \ldots, k$ with a fixed probability s_4 fold e_i and h_i^{-1}.

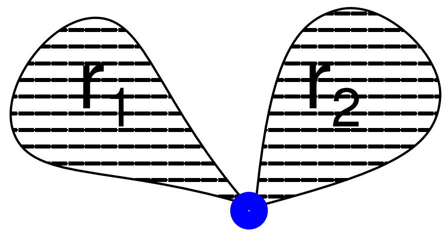
b) mark v, and add it to $M(D)$,

c) remove v from $A(D)$. Go to 5).

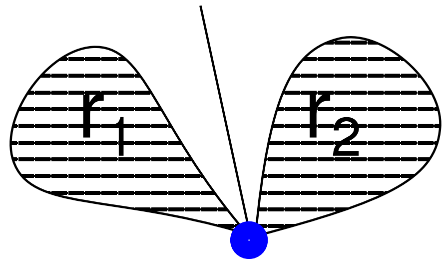
5) Denote the resulting diagram by $B_S(D)$. Output $B_S(D)$.

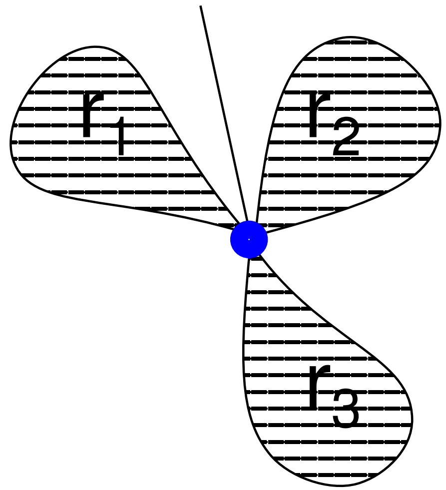
Random generator RG of van Kampen diagrams.


r_1

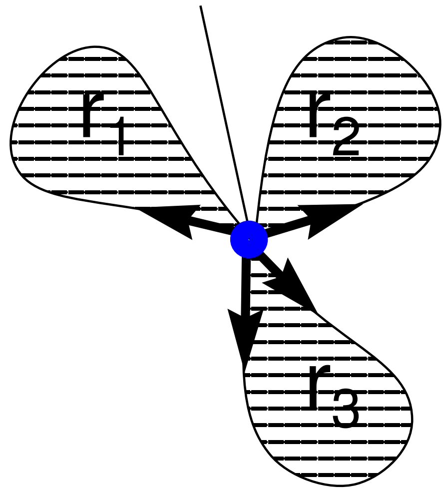


r1
r2

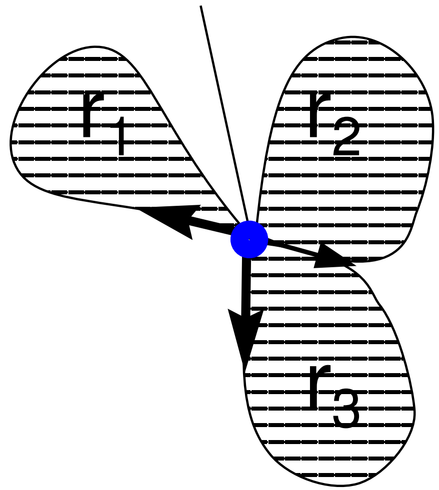

r_1
r_2



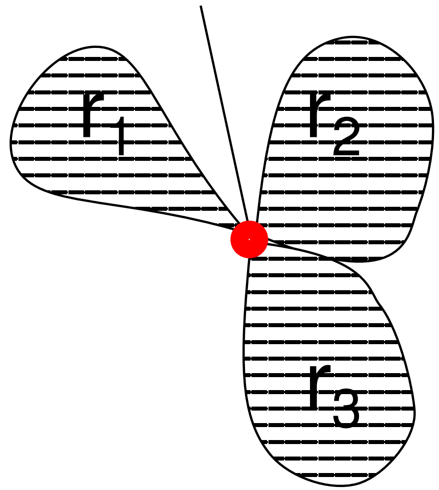
r1
r2
r3

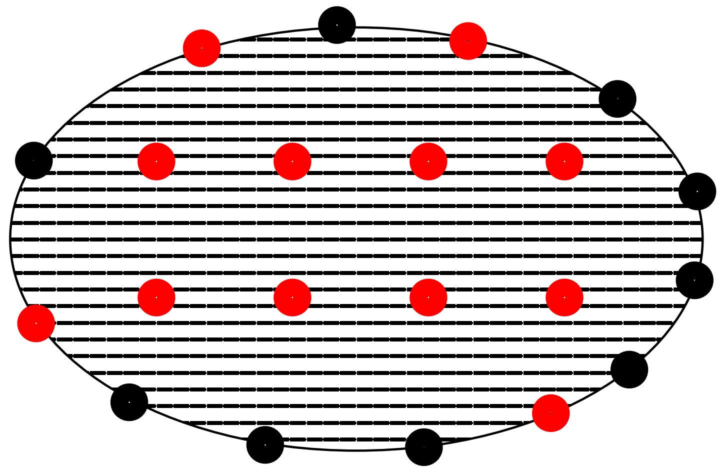


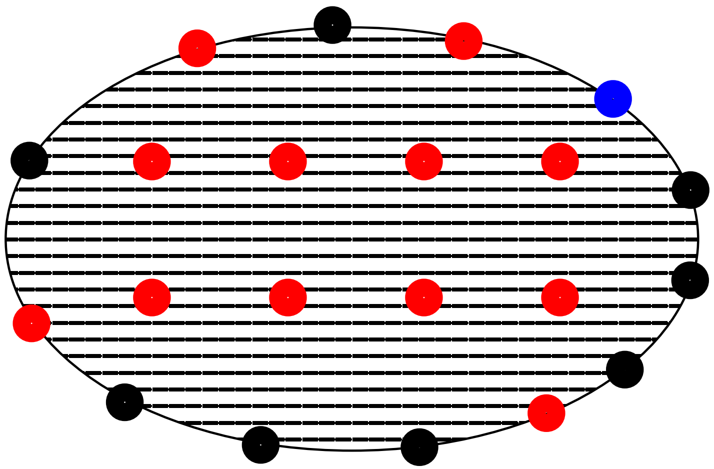
r_1
r_2
r_3

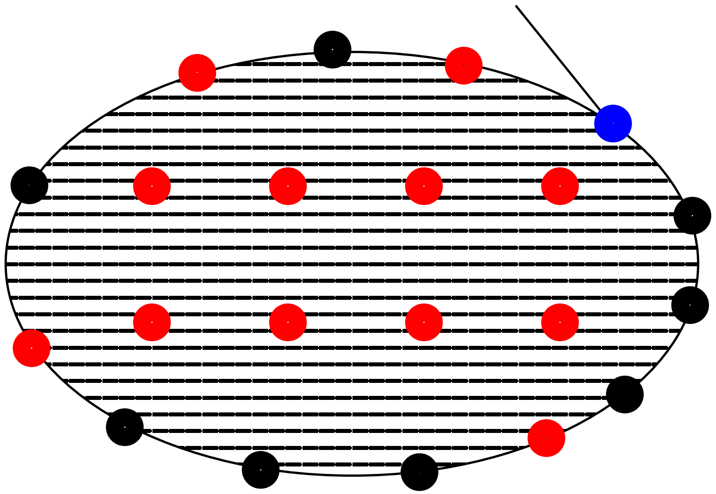


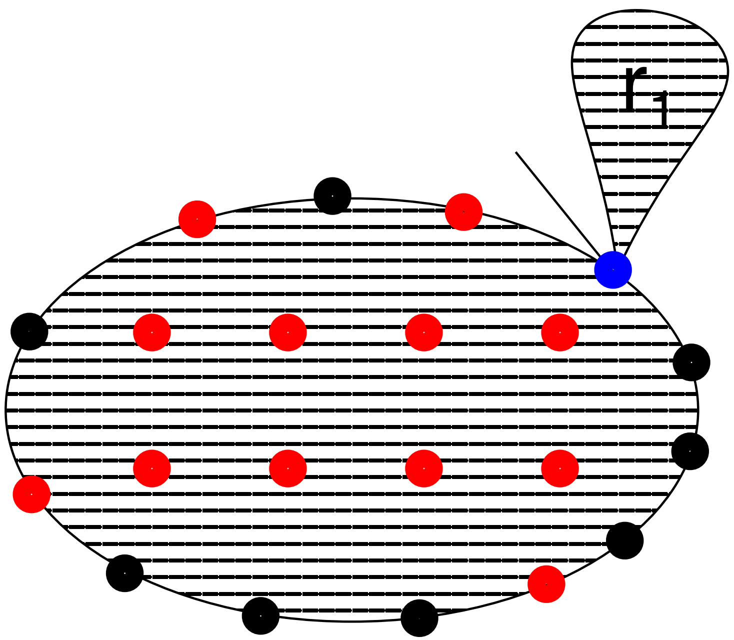
r1
r2
r3

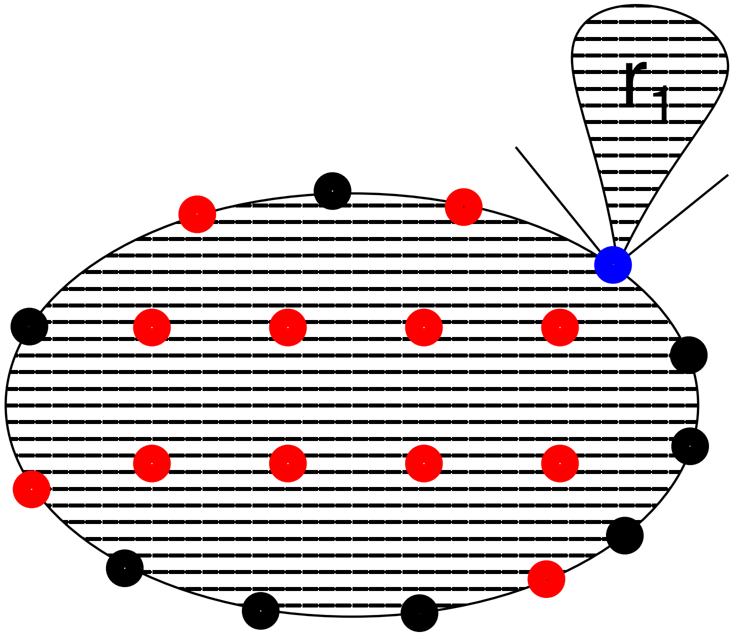

r1
r2
r3



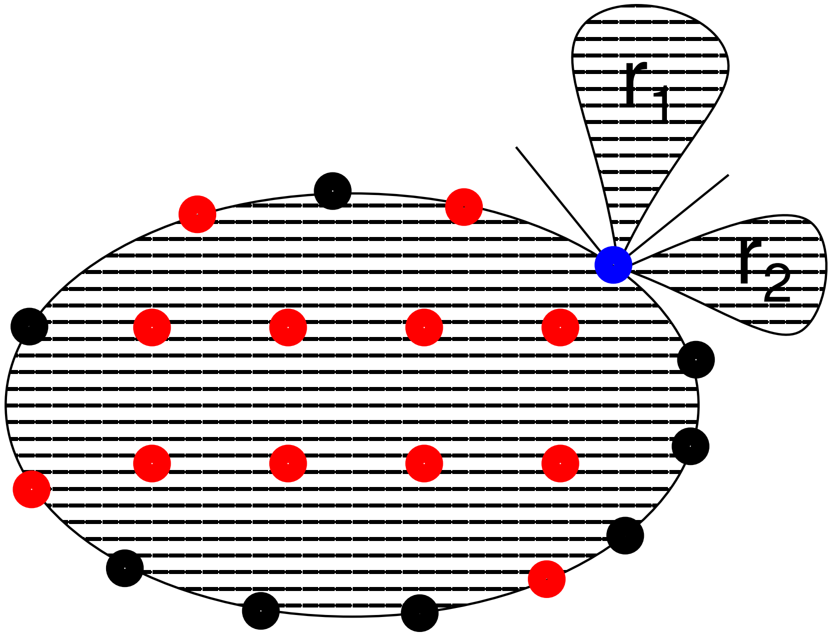


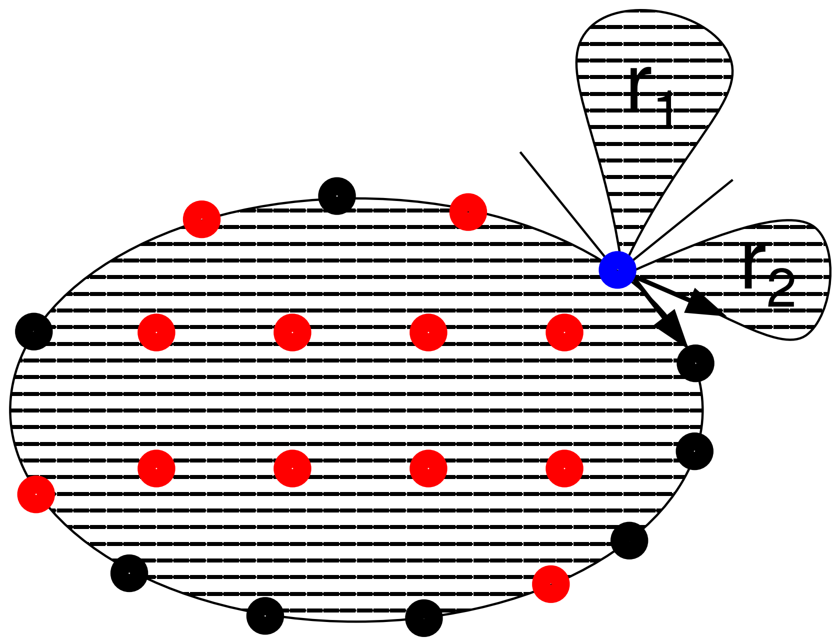


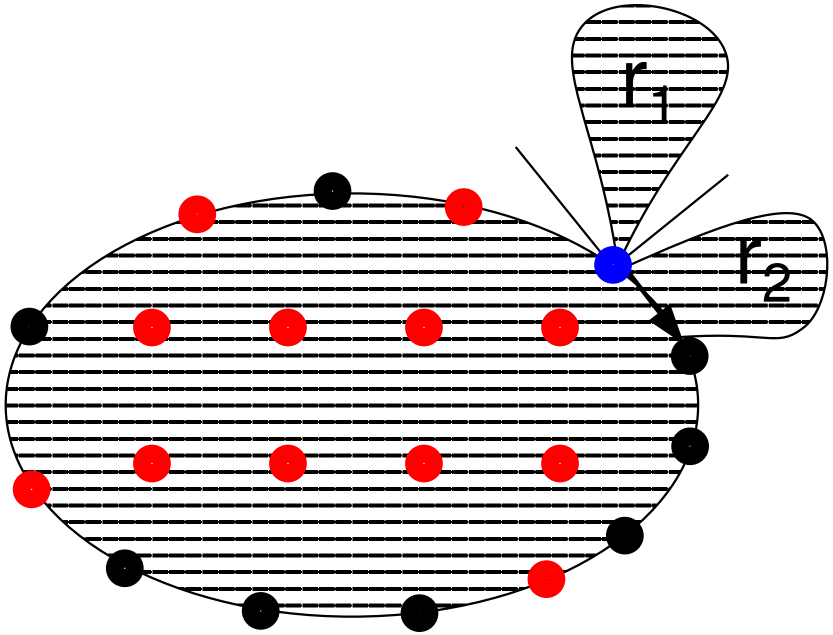

r1

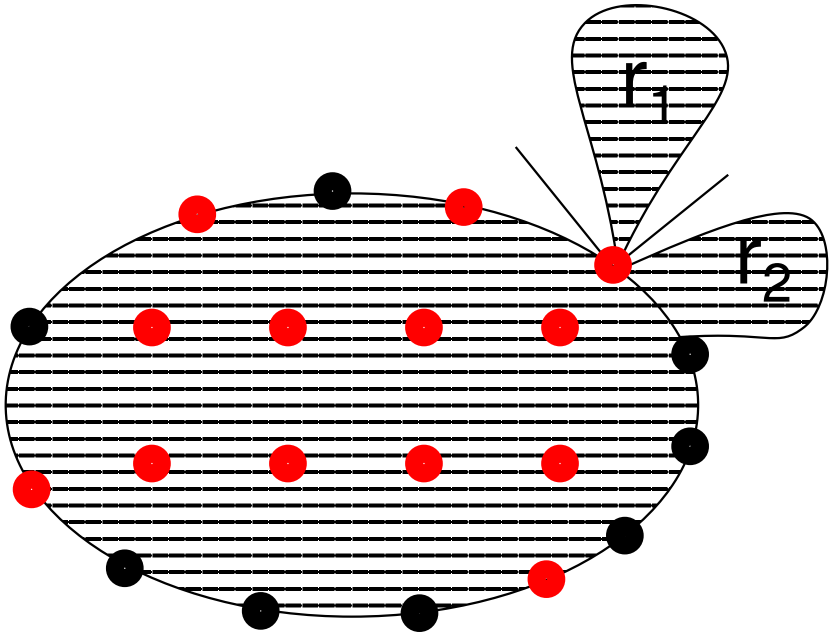


r_1


r1
r2


r_1
r_2


r1
r2



r_1
r_2