

On quotients over algebraic supergroups

A.N.Zubkov

1 Superschemes

Remind that a Z_2 -graded K -algebra A , where K is a ground field of zero or odd characteristic, is called *superalgebra*. A is called *commutative superalgebra* iff $ab = (-1)^{|a||b|}ba$ for all homogeneous elements $a, b \in A$ of parities $|a|, |b| \in Z_2$ respectively. Let $SAlg_K$ be a category of commutative superalgebras with even (parity preserving) morphisms. Denote by $K[m|n]$ a *free commutative superalgebra* with m even and n odd free generators.

A functor from $SAlg_K$ to the category of sets is called K -functor. The category of K -functors is denoted by \mathcal{F} . A K -functor $SSp R$, defined as $SSp R(A) = Hom_{SAlg_K}(R, A)$ for $A \in SAlg_K$, is called *affine superscheme*. The superalgebra $R \in SAlg_K$ is called *coordinate superalgebra* of the superscheme $SSp R$. If $X = SSp R$, then R is also denoted by $K[X]$.

Let X be an affine superscheme. A subfunctor $Y \subseteq X$ is said to be *closed (open)* iff there is an superideal $I \subseteq K[X]$ (respectively, an subset $I \subseteq K[X]$) such that $Y(A) = \{x \in X(A) | x(I) = 0\}$, (respectively, $Y(A) = \{x \in X(A) | \sum_{f \in I} Ax(f) = A\}$) for any $A \in SAlg_K$. If X is a K -functor, then its subfunctor $Y \subseteq X$ is called closed (open) iff for any $R \in SAlg_K$ and for any morphism of functors $p : SSp R \rightarrow X$ the subfunctor $p^{-1}(Y) \subseteq SSp R$ is closed (respectively, open).

A collection of open subfunctors $\{Y_j\}_{j \in J}$ of a K -functor X is called *open covering* iff for any field $A \in SAlg_K$ one has $X(A) = \bigcup_{j \in J} Y_j(A)$. Notice that an open subfunctor Y of X forms an open covering of X iff $Y = X$.

A K -functor X is said to be *local* if for any K -functor morphism $f : Y \rightarrow X$ and for any open covering $(Y_j)_{j \in J}$ of Y the diagram

$$(*) \text{Mor}_{\mathcal{F}}(Y, X) \xrightarrow{\alpha} \prod_{j \in J} \text{Mor}_{\mathcal{F}}(Y_j, X) \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{array} \prod_{j, j' \in J} \text{Mor}_{\mathcal{F}}(Y_j \cap Y_{j'}, X)$$

is exact. By definition,

$$\alpha(f) = (f|_{Y_j})_{j \in J}, \beta((f_j)_{j \in J}) = (f_j|_{Y_j \cap Y_{j'}})_{j, j' \in J}, \gamma((f_j)_{j \in J}) = (f_{j'}|_{Y_j \cap Y_{j'}})_{j, j' \in J}.$$

Proposition 1.1 *A K -functor is local iff for any $R \in SAlg_K$ and for arbitrary elements $f_1, \dots, f_n \in R_0$ such that $\sum R_{1 \leq i \leq n} R f_i = R$ the diagram*

$$(**) X(R) \rightarrow \prod_{1 \leq i \leq r} X(R_{f_i}) \rightrightarrows \prod_{1 \leq i, j \leq r} X(R_{f_i f_j})$$

is exact. All morphisms in $(**)$ are induced by the natural homomorphisms of superalgebras $R \rightarrow R_{f_i}, R_{f_i} \rightarrow R_{f_i f_j}, 1 \leq i, j \leq n$, in the same way as above.

Any affine superscheme is local. A K -functor X is called *superscheme* iff X is local and X has an open covering by affine superschemes. A closed/open subfunctor of a superscheme is again superscheme. The full subcategory of superschemes is denoted by \mathcal{SF} .

Remind that a superspace \mathcal{X} is a topological space $|\mathcal{X}|$ endowed with a sheaf of commutative superalgebras $\mathcal{O}_{\mathcal{X}}$ (in more general setting, superrings) whose any stalk $\mathcal{O}_{\mathcal{X},x}$ is a local superalgebra. A morphism of superspaces $(|\mathcal{X}|, \mathcal{O}_{\mathcal{X}}) \rightarrow (|\mathcal{Y}|, \mathcal{O}_{\mathcal{Y}})$ is a pair (f, f^*) , where $f : |\mathcal{X}| \rightarrow |\mathcal{Y}|$ is a morphism of topological spaces and $f^* : \mathcal{O}_{\mathcal{Y}} \rightarrow f_*\mathcal{O}_{\mathcal{X}}$ is a morphism of sheaves such that $f_x^* : \mathcal{O}_{\mathcal{Y},f(x)} \rightarrow \mathcal{O}_{\mathcal{X},x}$ is local for any $x \in |\mathcal{X}|$ (cf. [6, 7]). The category of superspaces is denoted by \mathcal{V} .

For $A \in \mathcal{SAlg}_K$ define a superspace $SSpec A$ as follows. The topological space $|SSpec A|$ coincides with the prime spectrum of A endowed with Zariski topology. For any open subset $U \subseteq |SSpec A|$ the superalgebra $\mathcal{O}_{SSpec A}(U)$ consists of all locally constant functions $f : U \rightarrow \bigsqcup_{P \in U} A_P$ such that $f(P) \in A_P = (A_0 \setminus P_0)^{-1}A, P \in U$. A superspace \mathcal{X} is called *Grothendieck superscheme* iff there is an open covering $|\mathcal{X}| = \bigcup_{i \in I} U_i$ such that $(U_i, \mathcal{O}_{\mathcal{X}|_{U_i}}) \simeq SSpec A_i, A_i \in \mathcal{SAlg}_K, i \in I$. The full subcategory of Grothendieck superschemes is denoted by \mathcal{SV} .

The proof of the following theorem is completely analogous to the proof of its classical counterpart (see [4, 6]).

Theorem 1.1 (*Comparison Theorem*) *There is an equivalence $\mathcal{SV} \rightarrow \mathcal{SF}$ defined by $\mathcal{X} \rightarrow X$, where $X(A) = Mor_{\mathcal{SV}}(SSpec A, \mathcal{X}), A \in \mathcal{SAlg}_K$. It takes open subschemes to open subfunctors.*

Example 1.1 *A supergrassmanian $Gr(r|s, m|n)$ is K -functor such that $Gr(r|s, m|n)(A)$ is the set of direct summands L of the free A -supermodule $A^{m|n}$, of fixed rank $r|s$. One can check that $Gr(r|s, m|n)$ is a superscheme (see [7, 6] for more details).*

2 Sheaves and sheafifications

Let R_1, \dots, R_n be a family of commutative R -superalgebras with respect to a set of morphisms $\iota_R^{R_i} : R \rightarrow R_i$ in \mathcal{SAlg}_K . Such a family is called *faithfully flat covering* of R (ff-covering, for short) whenever R -module $R_1 \times \dots \times R_n$ is faithfully flat. We say that R -superalgebra R' is *finitely presented* if $R' \simeq R[m|n]/I$, where $R[m|n] = R \otimes K[m|n]$ and $I \subseteq R[m|n]$ is a finitely generated superideal. Following [3] we call a ff-covering R_1, \dots, R_n *fppf-covering* if all R_i are finitely presented R -superalgebras.

A K -functor X is called *K -sheaf* if for any fppf-covering R_1, \dots, R_n of a superalgebra R the diagram

$$X(R) \rightarrow \prod_{1 \leq i \leq n} X(R_i) \rightrightarrows \prod_{1 \leq i, j \leq n} X(R_i \otimes_R R_j)$$

is exact, where the last two maps are induced by morphisms $R_i \rightarrow R_i \otimes_R R_j$ and $R_i \rightarrow R_j \otimes_R R_i$, respectively, defined as $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ for $a, b \in R_i$.

Proposition 2.1 *For any $X \in \mathcal{F}$ there are a K -sheaf \tilde{X} , called *sheafification* of X , and a morphism $p : X \rightarrow \tilde{X}$ such that for any K -sheaf Y the canonical map $Mor(\tilde{X}, Y) \rightarrow Mor(X, Y)$ induced by p is a bijection.*

This proposition is a partial case of more general statement about sheafifications of functors on sites (see [8]), Theorem 2.64).

Remark 2.1 *Assume that X commutes with finite direct products of superalgebras and for any fppf covering R' of a superalgebra R the induced map $X(R) \rightarrow X(R')$ is injective. Then p is an injection.*

Remark 2.2 *Using Theorem 1.1 one can prove that any superscheme is a K -sheaf.*

3 Quotients

Let G be an algebraic supergroup, that is G is an affine group K -functor whose coordinate ring $K[G]$ is a finitely generated commutative Hopf superalgebra [1]. Let H be a closed supersubgroup of G defined by a Hopf superideal $I_H \subseteq K[G]$. Denote by $K[G]^+$ a maximal ideal $\ker \epsilon_G$, where $\epsilon_G : K[G] \rightarrow K$ is a counit of Hopf superalgebra $K[G]$. Notice that $K[G]$ is a right $K[H]$ -supercomodule and $R = K[G]^H$ is a supersubalgebra of coinvariants.

The "naive" quotient of G over H is a K -functor $(G/H)_{(n)}(A) = G(A)/H(A)$, $A \in SAlg_K$. We define a quotient G/H as the sheafification of $(G/H)_{(n)}$. It can easily be checked that $(G/H)_{(n)}$ satisfies the conditions from Remark 2.1. (cf. [1, 3]). In particular, $(G/H)_{(n)}$ is a subfunctor of G/H .

Example 3.1 *Let V be a superspace of superdimension $m|n$. Let $GL(V)$ be a general linear supergroup. More precisely, for any $A \in SAlg_K$ $GL(V)(A)$ consists of all invertible even A -linear endomorphisms of $V \otimes A$. For a supersubspace $W \subseteq V$ denote by P_W the parabolic supersubgroup $Stab_{GL(V)}(W)$. One can prove that $GL(V)/P_W \simeq Gr(r|s, m|n)$, where $r|s$ is the superdimension of W .*

Let us formulate the following problems.

Problem 1 *Is any quotient G/H a superscheme?*

Problem 2 *When G/H is affine?*

By Proposition 6.3 from [1] there is an isomorphism of G onto a closed supersubgroup of $GL(V)$ for a suitable superspace V . Moreover, there exists a supersubspace $W \subseteq V$ such that H can be identified with $Stab_G(W)$. Thus G/H is identified with the sheafification of $(G/H)_{(n)} \subseteq GL(V)/P_W$, that is G/H is a subsheaf of the superscheme $Gr(r|s, m|n)$.

Problem 3 *Is G/H locally closed in $Gr(r|s, m|n)$?*

The following proposition is analogous to Takeuchi's theorem [11].

Proposition 3.1 *([1], Proposition 5.2 and Theorem 5.2) The following statements are equivalent :*

- 1) G/H is affine and isomorphic to $SSp R$;
- 2) $K[G]$ is a faithfully flat R -module and $I_H = K[G]R^+$, where $R^+ = K[G]^+ \cap R$;
- 3) The induction functor ind_H^G induces an equivalence between the category of left H -supermodules and a full subcategory of left G -supermodules.

The fundamental theorem of the algebraic group theory states that if G is an algebraic group and H is its closed normal subgroup, then G/H is again an algebraic group, see [4, 10, 9].

Theorem 3.1 ([1], Theorem 6.2) *If H is a closed normal supersubgroup of G , then G/H is an algebraic supergroup.*

Professor J.Brundan communicated me the following question. For an affine supergroup G denote by G_{ev} its *even* subgroup. More precisely, G_{ev} is a closed supersubgroup of G defined by the Hopf superideal $K[G]K[G]_1$. Assume that H_{ev} is a reductive group. Does it always imply that G/H is affine? Using some reduction to *infinitesimal* supersubgroups and Theorem 3.1 one can prove the following proposition.

Proposition 3.2 ([1], Proposition 8.1) *The answer for Brundan's question is positive whenever $\text{char}K = p > 0$.*

Following [4] we say that a morphism of K -functors $f : X \rightarrow Y$ is *affine* if, for any $R \in \text{SAlg}_K$ and any morphism $g : \text{SSp } R \rightarrow Y$, the fiber product $X \times_Y \text{SSp } R$ is an affine superscheme. If Y is an affine superscheme and $f : X \rightarrow Y$ is an affine morphism, then X is obviously affine. In fact, for $g = \text{id}_Y$ we have $X \simeq X \times_Y Y$.

Proposition 3.3 *If $f : X \rightarrow Y$ is affine and Y is a superscheme, then X is also a superscheme.*

An morphism of superschemes $(f, f^*) : \mathcal{X} \rightarrow \mathcal{Y}$ is called *faithfully flat* iff there are open coverings $\{\text{SSpec } A_i\}_{i \in I}$ and $\{\text{SSpec } B_j\}_{j \in J}$ of \mathcal{X} and \mathcal{Y} respectively such that for any $j \in J$ there is $i \in I$ with $f(\text{SSpec } A_i) \subseteq \text{SSpec } B_j$. Besides, A_i is a faithfully flat B_j -module with respect to f^* .

J.Brundan defines a quotient G/H as an affine and faithfully flat morphism of superschemes $\pi : G \rightarrow Y$ such that π is constant on H -cosets and for any morphism of superschemes $f : G \rightarrow Z$ which is also constant on H -cosets there is a unique morphism $g : Y \rightarrow Z$ with $f = g\pi$. We call Y a *Brundan's quotient*. Notice that the definition of a Brundan's quotient is not effective.

Proposition 3.4 (compare with [3]) *If a sheaf quotient G/H is a superscheme, then the canonical morphism $G \rightarrow G/H$ is affine and faithfully flat. In particular, it is also a Brundan's quotient.*

The proof of this proposition can be copied from [3], part I, (5.7). The next proposition plays the crucial role in its proof (see [4], III, §1, Corollary 2.12).

Proposition 3.5 *Let*

$$\begin{array}{ccc} X & \rightarrow & Y \\ \downarrow & & \downarrow \\ Z & \rightarrow & U \end{array}$$

be a cartesian square in the category of sheaves. Assume that $Z \rightarrow U$ is an epimorphism of sheaves and $X \rightarrow Z$ is affine. Then $Y \rightarrow U$ is also affine.

Corollary 3.1 *Let G be an algebraic supergroup and $H_1 \leq H_2$ are closed supersubgroups of G . If G/H_2 is a superscheme and H_2/H_1 is an affine superscheme, then G/H_1 is a superscheme.*

In fact, the following diagram

$$\begin{array}{ccc} G \times H_2/H_1 & \rightarrow & G/H_1 \\ \downarrow & & \downarrow \\ G & \rightarrow & G/H_2 \end{array},$$

where $G \times H_2/H_1 \rightarrow G$ is the canonical projection and $G \times H_2/H_1 \rightarrow G/H_1$ is induced by $(g, hH_1(A)) \rightarrow ghH_1(A), g \in G(A), h \in H_2(A), A \in SAlg_K$, is a cartesian square. By Proposition 3.4 the canonical morphism $G/H_1 \rightarrow G/H_2$ is affine. It remains to apply Proposition 3.3.

Proposition 3.6 *If $\text{char}K = p > 0$, then G/H is a superscheme.*

The following is a sketch of the proof. Denote by $G^{(1)}$ the first infinitesimal supersubgroup of G . Remind that $K[G]_0^p$ is an even Hopf supersubalgebra of $K[G]$. The inclusion $K[G]_0^p \rightarrow K[G]$ induces an epimorphism of supergroups $G \rightarrow SSp K[G]_0^p$ whose kernel coincides with $G^{(1)}$. The supergroup $G^{(1)}$ is finite. Since $G/HG^{(1)}$ is a scheme (therefore, an even superscheme) and $HG^{(1)}/H \simeq G^{(1)}/G^{(1)} \cap H$ is affine by Theorem 7.1 from [1], we are done by Corollary 3.1.

The problems 1 and 2 can be generalized as follows. Replace G by an affine superscheme X and assume that H acts on X freely, that is for any $A \in SAlg_K$ and for any $x \in X(A)$ the stabilizer $Stab_G(x) = \{g \in G(A) | xg = x\}$ is trivial.

Proposition 3.7 *(see [2]) Let H be a finite supergroup acting on X freely. Then X/H is again affine. In particular, $X/H \simeq SSp K[X]^H$. Moreover, $K[X]$ is a finitely generated projective $K[X]^H$ -module.*

Remark 3.1 *Observe that Corollary 3.1 can be generalized as follows. Assume that G acts on an affine superscheme X freely. If H is a closed supersubgroup of G such that G/H is affine and X/G is a superscheme, then X/H is also superscheme.*

References

- [1] A.N.Zubkov, *Affine quotients of supergroups*, arXiv : math/0804.3493v2.
- [2] A.N.Zubkov, *On quotients of affine superschemes over finite supergroups*, arXiv : math/0901.4687.
- [3] J.Jantzen, *Representations of algebraic groups*, Academic Press, Inc., 1987.
- [4] M.Demazure and P.Gabriel, *Groupes algebriques I*, Paris/Amsterdam, 1970.
- [5] N.Bourbaki, *Commutative Algebra*, Addison-Wesley, 1972.
- [6] L.Caston and R.Fioresi, *Mathematical foundations of supersymmetry*, arXiv : math/0710.5742v1.
- [7] Yu.I.Manin, *Gauge field theory and complex geometry*, Springer, 1988.
- [8] A.Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, arXiv : math/0412512v4 .

- [9] W.C.Waterhouse, *Introduction to affine group schemes*, Springer-Verlag, 1979.
- [10] J.E.Humphreys, *Linear algebraic groups*, Springer-Verlag, 1975.
- [11] M.Takeuchi, *A correspondence between Hopf ideals and sub-Hopf algebras*, Manuscripta Math., 7(1972), 251-270.