On quotients over algebraic supergroups

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1 Superschemes

Remind that a Z_2-graded K-algebra A, where K is a ground field of zero or odd characteristic, is called *superalgebra*. A is called *commutative* superalgebra iff $ab = (-1)^{|a||b|}ba$ for all homogeneous elements $a, b \in A$ of parities $|a|, |b| \in Z_2$ respectively. Let $SAlg_K$ be a category of commutative superalgebras with even (parity preserving) morphisms. Denote by $K[m|n]$ a *free* commutative superalgebra with m even and n odd free generators.

A functor from $SAlg_K$ to the category of sets is called K- functor. The category of K-functors is denoted by $\mathcal{F}$. A K-functor $SSp\ R$, defined as $SSp\ R(A) = Hom_{SAlg_K}(R, A)$ for $A \in SAlg_K$, is called *affine superscheme*. The superalgebra $R \in SAlg_K$ is called *coordinate superalgebra* of the superscheme $SSp\ R$. If $X = SSp\ R$, then R is also denoted by $K[X]$.

Let X be an affine superscheme. A subfunctor $Y \subseteq X$ is said to be *closed (open)* iff there is an superideal $I \subseteq K[X]$ (respectively, an subset $I \subseteq K[X]$) such that $Y(A) = \{x \in X(A) | x(I) = 0\}$, (respectively, $Y(A) = \{x \in X(A) | \sum_{f \in I} Ax(f) = A\}$) for any $A \in SAlg_K$. If X is a K-functor, then its subfunctor $Y \subseteq X$ is called closed (open) iff for any $R \in SAlg_K$ and for any morphism of functors $p : SSp\ R \to X$ the subfunctor $p^{-1}(Y) \subseteq SSp\ R$ is closed (respectively, open).

A collection of open subfunctors $\{Y_j\}_{j \in J}$ of a K-functor X is called *open covering* iff for any field $A \in SAlg_K$ one has $X(A) = \bigcup_{j \in J} Y_j(A)$. Notice that an open subfunctor Y of X forms an open covering of X iff $Y = X$.

A K-functor X is said to be *local* if for any K-functor morphism $f : Y \to X$ and for any open covering $(Y_j)_{j \in J}$ of Y the diagram

$$(*)\ Mor_{\mathcal{F}}(Y, X) \stackrel{\alpha}{\to} \prod_{j \in J} Mor_{\mathcal{F}}(Y_j, X) \begin{array}{c}\stackrel{\beta}{\to} \\ \stackrel{\gamma}{\to}\end{array} \prod_{j, j' \in J} Mor_{\mathcal{F}}(Y_j \cap Y_{j'}, X)$$

is exact. By definition,

$$\alpha(f) = (f|_{Y_j})_{j \in J}, \beta((f_j)_{j \in J}) = (f_j|_{Y_j \cap Y_{j'}})_{j, j' \in J}, \gamma((f_j)_{j \in J}) = (f_{j'}|_{Y_j \cap Y_{j'}})_{j, j' \in J}.$$

Proposition 1.1 *A K-functor is local iff for any $R \in SAlg_K$ and for arbitrary elements $f_1, \ldots, f_n \in R_0$ such that $\sum R_{1 \leq i \leq n} R f_i = R$ the diagram*

$$(**)\ X(R) \to \prod_{1 \leq i \leq r} X(R_{f_i}) \begin{array}{c}\to \\ \to\end{array} \prod_{1 \leq i, j \leq r} X(R_{f_i f_j})$$

is exact. All morphisms in (**) are induced by the natural homomorphisms of superalgebras $R \to R_{f_i}, R_{f_i} \to R_{f_i f_j}, 1 \leq i, j \leq n$, in the same way as above.

Any affine superscheme is local. A K-functor X is called *superscheme* iff X is local and X has an open covering by affine superschemes. A closed/open subfunctor of a superscheme is again superscheme. The full subcategory of superschemes is denoted by $\mathcal{SF}$.

Remind that a superspace $\mathcal{X}$ is a topological space $|\mathcal{X}|$ endowed with a sheaf of commutative superalgebras $\mathcal{O}_{\mathcal{X}}$ (in more general setting, superrings) whose any stalk $\mathcal{O}_{\mathcal{X},x}$ is a local superalgebra. A morphism of superspaces $(|\mathcal{X}|, \mathcal{O}_{\mathcal{X}}) \to (|\mathcal{Y}|, \mathcal{O}_{\mathcal{Y}})$ is a pair (f, f^*), where $f : |\mathcal{X}| \to |\mathcal{Y}|$ is a morphism of topological spaces and $f^* : O_{\mathcal{Y}} \to f_* O_{\mathcal{X}}$ is a morphism of sheaves such that $f_x^* : O_{\mathcal{Y},f(x)} \to O_{\mathcal{X},x}$ is local for any $x \in |\mathcal{X}|$ (cf. [6, 7]). The category of superspaces is denoted by $\mathcal{V}$.

For $A \in SAlg_K$ define a superspace $SSpec\ A$ as follows. The topological space $|SSpec\ A|$ coincides with the prime spectrum of A endowed with Zariski topology. For any open subset $U \subseteq |SSpec\ A|$ the superalgebra $\mathcal{O}_{SSpec\ A}(U)$ consists of all locally constant functions $f : U \to \bigsqcup_{P \in U} A_P$ such that $f(P) \in A_P = (A_0 \setminus P_0)^{-1}A, P \in U$. A superspace $\mathcal{X}$ is called *Grothendieck superscheme* iff there is an open covering $|\mathcal{X}| = \bigcup_{i \in I} U_i$ such that $(U_i, \mathcal{O}_{\mathcal{X}}|_{U_i}) \simeq SSpec\ A_i, A_i \in SAlg_K, i \in I$. The full subcategory of Grothendieck superschemes is denoted by $\mathcal{SV}$.

The proof of the following theorem is completely analogous to the proof of its classical counterpart (see [4, 6]).

Theorem 1.1 *(Comparison Theorem) There is an equivalence $\mathcal{SV} \to \mathcal{SF}$ defined by $\mathcal{X} \to X$, where $X(A) = Mor_{\mathcal{SV}}(SSpec\ A, \mathcal{X}), A \in SAlg_K$. It takes open subschemes to open subfunctors.*

Example 1.1 *A supergrassmanian $Gr(r|s, m|n)$ is K-functor such that $Gr(r|s, m|n)(A)$ is the set of direct summands L of the free A-supermodule $A^{m|n}$, of fixed rank $r|s$. One can check that $Gr(r|s, m|n)$ is a superscheme (see [7, 6] for more details).*

2 Sheaves and sheafifications

Let $R_1, \ldots, R_n$ be a family of commutative R-superalgebras with respect to a set of morphisms $\iota_R^{R_i} : R \to R_i$ in $SAlg_K$. Such a family is called *faithfully flat covering* of R (ff-covering, for short) whenever R-module $R_1 \times \ldots \times R_n$ is faithfully flat. We say that R-superalgebra R' is *finitely presented* if $R' \simeq R[m|n]/I$, where $R[m|n] = R \otimes K[m|n]$ and $I \subseteq R[m|n]$ is a finitely generated superideal. Following [3] we call a ff-covering $R_1, \ldots, R_n$ *fppf-covering* if all R_i are finitely presented R-superalgebras.

A K-functor X is called *K-sheaf* if for any fppf-covering $R_1, \ldots R_n$ of a superalgebra R the diagram

$$X(R) \to \prod_{1 \leq i \leq n} X(R_i) \begin{array}{c}\to\\ \to\end{array} \prod_{1 \leq i,j \leq n} X(R_i \otimes_R R_j)$$

is exact, where the last two maps are induced by morphisms $R_i \to R_i \otimes_R R_j$ and $R_i \to R_j \otimes_R R_i$, respectively, defined as $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ for $a, b \in R_i$.

Proposition 2.1 *For any $X \in \mathcal{F}$ there are a K-sheaf $\tilde{X}$, called sheafification of X, and a morphism $p : X \to \tilde{X}$ such that for any K-sheaf Y the canonical map $Mor(\tilde{X}, Y) \to Mor(X, Y)$ induced by p is a bijection.*

This proposition is a partial case of more general statement about sheafifications of functors on sites (see [8]), Theorem 2.64).

Remark 2.1 *Assume that X commutes with finite direct products of superalgebras and for any fppf covering R' of a superalgebra R the induced map $X(R) \to X(R')$ is injective. Then p is an injection.*

Remark 2.2 *Using Theorem 1.1 one can prove that any superscheme is a K-sheaf.*

3 Quotients

Let G be an algebraic supergoup, that is G is an affine group K-functor whose coordinate ring $K[G]$ is a finitely generated commutative Hopf superalgebra [1]. Let H be a closed supersubgroup of G defined by a Hopf superideal $I_H \subseteq K[G]$. Denote by $K[G]^+$ a maximal ideal $\ker \epsilon_G$, where $\epsilon_G : K[G] \to K$ is a counit of Hopf superalgebra $K[G]$. Notice that $K[G]$ is a right $K[H]$-supercomodule and $R = K[G]^H$ is a supersubalgebra of coinvariants.

The *"naive"* quotient of G over H is a K-functor $(G/H)_{(n)}(A) = G(A)/H(A), A \in SAlg_K$. We define a quotient G/H as the sheafification of $(G/H)_{(n)}$. It can easily be checked that $(G/H)_{(n)}n$ satisfies the conditions from Remark 2.1. (cf. [1, 3]). In particular, $(G/H)_{(n)}$ is a subfunctor of G/H.

Example 3.1 *Let V be a superspace of superdimension $m|n$. Let $GL(V)$ be a general linear supergroup. More precisely, for any $A \in SAlg_K$ $GL(V)(A)$ consists of all invertible even A-linear endomorphisms of $V \otimes A$. For a supersubspace $W \subseteq V$ denote by P_W the parabolic supersubgoup $Stab_{GL(V)}(W)$. One can prove that $GL(V)/P_W \simeq Gr(r|s, m|n)$, where $r|s$ is the superdimension of W.*

Let us formulate the following problems.

Problem 1 *Is any quotient G/H a superscheme?*

Problem 2 *When G/H is affine?*

By Proposition 6.3 from [1] there is an isomorphism of G onto a closed supersubgroup of $GL(V)$ for a suitable superspace V. Moreover, there exists a supersubspace $W \subseteq V$ such that H can be identified with $Stab_G(W)$. Thus G/H is identified with the sheafification of $(G/H)_{(n)} \subseteq GL(V)/P_W$, that is G/H is a subsheaf of the superscheme $Gr(r|s, m|n)$.

Problem 3 *Is G/H locally closed in $Gr(r|s, m|n)$?*

The following proposition is analogous to Takeuchi's theorem [11].

Proposition 3.1 *([1], Proposition 5.2 and Theorem 5.2) The following statements are equivalent :*
1) G/H is affine and isomorphic to $SSp\ R$;
2) $K[G]$ is a faithfully flat R-module and $I_H = K[G]R^+$, where $R^+ = K[G]^+ \bigcap R$;
3) The induction functor ind_H^G induces an equivalence between the category of left H-supermodules and a full subcategory of left G-supermodules.

The fundamental theorem of the algebraic group theory states that if G is an algebraic group and H is its closed normal subgroup, then G/H is again an algebraic group, see [4, 10, 9].

Theorem 3.1 *([1], Theorem 6.2) If H is a closed normal supersubgroup of G, then G/H is an algebraic supergroup.*

Professor J.Brundan communicated me the following question. For an affine supergroup G denote by G_{ev} its *even* subgroup. More precisely, G_{ev} is a closed supersubgroup of G defined by the Hopf superideal $K[G]K[G]_1$. Assume that H_{ev} is a reductive group. Does it always imply that G/H is affine? Using some reduction to *infinitesimal* supersubgroups and Theorem 3.1 one can prove the following proposition.

Proposition 3.2 *([1], Proposition 8.1) The answer for Brundan's question is positive whenever $char K = p > 0$.*

Following [4] we say that a morphism of K-functors $f : X \to Y$ is *affine* if, for any $R \in SAlg_K$ and any morphism $g : SSp\ R \to Y$, the fiber product $X \times_Y SSp\ R$ is an affine superscheme. If Y is an affine superscheme and $f : X \to Y$ is an affine morphism, then X is obviously affine. In fact, for $g = id_Y$ we have $X \simeq X \times_Y Y$.

Proposition 3.3 *If $f : X \to Y$ is affine and Y is a superscheme, then X is also a superscheme.*

An morphism of superschemes $(f, f^*) : \mathcal{X} \to \mathcal{Y}$ is called faithfully flat iff there are open coverings $\{SSpec\ A_i\}_{i \in I}$ and $\{SSpec\ B_j\}_{j \in J}$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively such that for any $j \in J$ there is $i \in I$ with $f(SSpec\ A_i) \subseteq SSpec B_j$. Besides, A_i is a faithfully flat B_j-module with respect to f^*.

J.Brundan defines a quotient G/H as an affine and faithfully flat morphism of superschemes $\pi : G \to Y$ such that π is constant on H-cosets and for any morphism of superschemes $f : G \to Z$ which is also constant on H-cosets there is a unique morphism $g : Y \to Z$ with $f = g\pi$. We call Y a *Brundan's quotient*. Notice that the definition of a Brundan's quotient is not effective.

Proposition 3.4 *(compare with [3]) If a sheaf quotient G/H is a superscheme, then the canonical morphism $G \to G/H$ is affine and faithfully flat. In particular, it is also a Brundan's quotient.*

The proof of this proposition can be copied from [3], part I, (5.7). The next proposition plays the crucial role in its proof (see [4], III, §1, Corollary 2.12).

Proposition 3.5 *Let*

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ Z & \to & U \end{array}$$

be a cartesian square in the category of sheaves. Assume that $Z \to U$ is an epimorphism of sheaves and $X \to Z$ is affine. Then $Y \to U$ is also affine.

Corollary 3.1 *Let G be an algebraic supergroup and $H_1 \leq H_2$ are closed supersubgroups of G. If G/H_2 is a superscheme and H_2/H_1 is an affine superscheme, then G/H_1 is a superscheme.*

In fact, the following diagram

$$\begin{array}{ccc} G\times H_2/H_1 & \to & G/H_1 \\ \downarrow & & \downarrow \\ G & \to & G/H_2 \end{array},$$

where $G\times H_2/H_1 \to G$ is the canonical projection and $G\times H_2/H_1\to G/H_1$ is induced by $(g, hH_1(A))\to ghH_1(A), g\in G(A), h\in H_2(A), A\in SAlg_K$, is a cartesian square. By Proposition 3.4 the canonical morphism $G/H_1\to G/H_2$ is affine. It remains to apply Proposition 3.3.

Proposition 3.6 *If* $charK = p > 0$*, then* G/H *is a superscheme.*

The following is a sketch of the proof. Denote by $G^{(1)}$ the first infinitesimal supersubgroup of G. Remind that $K[G]_0^p$ is an even Hopf supersubalgebra of $K[G]$. The inclusion $K[G]_0^p \to K[G]$ induces an epimorphism of supergroups $G\to SSp\ K[G]_0^p$ whose kernel coincides with $G^{(1)}$. The supergroup $G^{(1)}$ is finite. Since $G/HG^{(1)}$ is a scheme (therefore, an even superscheme) and $HG^{(1)}/H\simeq G^{(1)}/G^{(1)}\bigcap H$ is affine by Theorem 7.1 from [1], we are done by Corollary 3.1.

The problems 1 and 2 can be generalized as follows. Replace G by an affine superscheme X and assume that H acts on X freely, that is for any $A\in SAlg_K$ and for any $x\in X(A)$ the stabilizer $Stab_G(x)=\{g\in G(A)|xg=x\}$ is trivial.

Proposition 3.7 *(see [2]) Let* H *be a finite supergroup acting on* X *freely. Then* X/H *is again affine. In particular,* $X/H\simeq SSp\ K[X]^H$*. Moreover,* $K[X]$ *is a finitely generated projective* $K[X]^H$*-module.*

Remark 3.1 *Observe that Corollary 3.1 can be generalized as follows. Assume that* G *acts on an affine superscheme* X *freely. If* H *is a closed supersubgroup of* G *such that* G/H *is affine and* X/G *is a superscheme, then* X/H *is also superscheme.*

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