Automorphisms of partially commutative nilpotent R-groups.

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Contents

Contents

Binomial ring

A.G. Miasnikov, V.N. Remeslennikov. *Isomorphisms and elementary properties of nilpotent powered groups* (1981)

Definition

R is called a binomial ring, if R is Abelian domain of integrity, R contains $\mathbb{Z}$ as subring and for any $\lambda \in R$ and $n \in \mathbb{N}$, binomial coefficient

$$C_\lambda^n = \frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}{n!}$$

contains in R.

Examples: Z, Q, field of zero characterisitic, ring of polinoms over field of zero characteristic.

Nilpotent R-groups

A nilpotent group G of nilpotency class m is called a R-group if for any $x \in G$ and $\lambda \in R$ there is a uniquely defined element $x^\lambda \in G$ and the following axioms are satisfied $(x, y, x_1, \ldots, x_n \in G, \lambda, \mu \in R)$:

1. $x^1 = x,\ x^\lambda x^\mu = x^{\lambda+\mu},\ (x^\lambda)^\mu = x^{\lambda\mu}$.
2. $y^{-1}x^\lambda y = (y^{-1}xy)^\lambda$.
3. $x_1^\lambda \ldots x_n^\lambda = (x_1, \ldots x_n)^\lambda \tau_2^{C_\lambda^2}(X) \ldots \tau_m^{C_\lambda^m}(X)$, where $X = \{x_1, \ldots, x_n\}$, $\tau_i(X)$ – i-th Petresco word. Recall that for each $k \in \mathbb{N}$, k-th Petrsco word is defined recursively by the relation:

$$x_1^i \ldots x_n^i = \tau_1^{C_i^1}(X)\tau_2^{C_i^2}(X) \ldots \tau_{i-1}^{C_i^{i-1}}(X)\tau_i^{C_i^i}(X)$$

in the free group F with basis $x_1, \ldots, x_n$. For example,
$\tau_1(X) = x_1x_2 \ldots x_n,\ \tau_2(X) = \prod_{i<j,\ i,j=1}^{n} [x_i, x_j]\ mod\ \gamma_3(F)$, where
$\gamma_3(F)$ – third term of the lower central series of group F.

Nilpotency class 2

We consider the case of nilpotent class 2 groups, i.e $m = 2$ in the definition above. So, the axiom (3) looks in the following way

$3'$. $x_1^\lambda \dots x_n^\lambda = (x_1, \dots x_n)^\lambda \tau_2^{C_\lambda^2}(X)$, *where* $\tau_2(x_1, \dots, x_n) = \prod_{i<j,\ i,j=1}^{n} [x_i, x_j]$.

Definition

$$G \in N_2 \text{ if } \forall x, y, z \in G\ [x, y, z] = [[x, y], z] = 1.$$

Denote by $N_{2,R}$ the variety of nilpotent class 2 R-groups.

Finally, define partially commutative nilpotent group in the variety $N_{2,R}$:

$$G_\Gamma = \langle X | R_\Gamma \rangle_{N_{2,R}},$$

where $R_\Gamma = \{[x_i, x_j] = 1 | \forall (x_i, x_j) \in E(\Gamma)\}\rangle_{N_{2,R}}$.

Contents

Normal form

Proposition

1. *Quotient group* $\overline{G_\Gamma} = G_\Gamma / G'_\Gamma$ *has linear vector space structure over* R *with basis* $x_1, \ldots, x_n$.
2. *Commutant* G'_Γ *has linear vector space structure over* $\mathbb{R}$ *with basis* $y_{ij} = [x_i, x_j]$, *where* $y_{ij} = [x_i, x_j] \neq 1$ *in* G_Γ *and* $i < j$.
3. *Any element* g *of* G_Γ *can be uniquely presented in the following way*

$$g = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \prod y_{kl}^{\beta_{kl}}, \tag{1}$$

where $x_i \in X$, $y_{kl} = [x_k, x_l] \neq 1,\ k < l$, *and* $\alpha_i, \beta_{kl} \in \mathbb{R}$.

Contents

Compressed graph

A.J. Duncan, I.V. Kazachkov, V.N. Remeslennikov. *Orthogonal systems in finite graphs*

For any $x, y \in X$ define distance $d(x, y)$ as minimum of all path length's joining x and y. And $d(x, y) = \infty$ if x, y aren't connected.

Definition

Let $x \in X$, define as $x^{\perp} = \{y \in X \mid d(x, y) \leq 1\}$. Let $x \in X$, define as $x^{o} = \{y \in X \mid d(x, y) = 1\}$

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Definition

We write $x \sim_\perp y$, iff $x^\perp = y^\perp$, and $x \sim_o y$, iff $x^o = y^o$. Finally, $x \sim y$ iff $x \sim_o y$ or $x \sim_\perp y$.

$\sim$ - is equivalence relation on X. Let $[x] = \{y \in X | x \sim y\}$.
Denote by Γ^{comp} compressed graph with vertices set $X^{comp} = \{[x] | x \in X\}$ and vertices $[x]$ and $[y]$ are joint iff, x and y joint in the Γ.

Theorem (A.J. Duncan, I.V. Kazachkov, V.N. Remeslennikov)

$Aut(\Gamma)$ has the following decomposition:

$$Aut(\Gamma) = (\prod_{[x] \in X^c} S_{|[x]|}) \leftthreetimes Aut(\Gamma^c).$$

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Contents

Partial order on X

A.J. Duncan, I.V. Kazachkov, V.N. Remeslennikov.

Definition

For any $x \in X$, denote by $ad(x) = (x^{\perp} \setminus x)^{\perp}$.

Lemma

Let $x, y \in X$, then the following holds:

1) *if $y \in ad(x)$, then $ad(y) \subseteq ad(x)$;*
2) *for any $x \in X$, $[x] \subseteq ad(x)$;*
3) *$ad(x) = ad(y)$ iff $[x] = [y]$;*
4) *for any $s, t \in X$ such that $s \in ad(x), t \in ad(y)$, if $[x, y] = 1$, then $[s, t] = 1$.*

Define partial order on X: we say that $x <_{ad} y$ iff $ad(x) \subsetneq ad(y)$. We say that $x \leq_{ad} y$ iff $ad(x) \subseteq ad(y)$

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Authomorphisms of free partially commutative groups.

- M.R.Laurence. *A generating set for the automorphism group of a graph group*
- A.J. Duncan, I.V. Kazachkov, V.N. Remeslennikov. *Authomorphisms of Partially Commutative Groups*
- G.A. Noskov. *The image of the authomorphism group of a graph group under the abelinization map*
- Other works.

Contents

R-authomorphisms

We define the map ϕ on generating set $X = \{x_1, \dots x_n\}$:

$$\begin{aligned} \phi(x_1) &= x_1^{\alpha_{11}} \dots x_n^{\alpha_{1n}} c_1, \\ &\dots \\ \phi(x_n) &= x_1^{\alpha_{n1}} \dots x_n^{\alpha_{nn}} c_n, \end{aligned} \tag{2}$$

where $\alpha_{i,j} \in R$, and $c_i \in G_\Gamma'$.

Definition

Let $G \in N_{2,R}$. The map $\phi : G \mapsto G$ is called R-authomorpism, if

1) *ϕ – group authomorphism;*
2) *for any $x \in G$ and $\alpha \in R$ holds $\phi(g^\alpha) = \phi(g)^\alpha$.*

Theorem

Let $G_\Gamma \in N_{2,R}$ with generating set $X = \{x_1, \ldots, x_n\}$. Then the following holds:

1) *exists shortly exact sequence:*

$$1 \mapsto IAut(G_\Gamma) \mapsto Aut(G_\Gamma) \stackrel{f}{\mapsto} GL(n, R) \mapsto 1,$$

where f – facrorization homomorphism, $IAut(G_\Gamma) = \ker f$, $Aut_l(G_\Gamma) = Imf$ – subgroup of factor authomorphisms in $GL(n, R)$;

2) *$IAut(G_\Gamma)$ – abelian normal subgroup, isomorphic to $\underbrace{G'_\Gamma \times \ldots \times G'_\Gamma}_{n \text{ раз}}$;*

3) *generating set of $Aut(G_\Gamma)$ is union of generating set for $Aut_l(G_\Gamma)$ and generating set for $IAut(G_\Gamma)$.*

Contents

Criterion to be an authomorphism

Theorem

The map $\phi : G_\Gamma \mapsto G_\Gamma$ is R-authomorphism iff the next conditions holds:

1. *Matrix $[\theta] = (\alpha_{ij}),\ i, j = 1, \ldots n$ is Γ - admissible matrix.*
2. *Column $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is any element of free R-module $(G'_\Gamma)^n$.*

Theorem

1. *The projection map* $\pi : Aut_l(G_\Gamma) \mapsto Aut(\Gamma^c)$ *is epimorphism. Let* $\ker \pi = Aut_l^0(G_\Gamma)$. *Then* $Aut_l(G_\Gamma) = Aut_l^0(G_\Gamma) \leftthreetimes Aut(\Gamma^c)$.
2. *Matrices from* $Aut_l^0(G_\Gamma)$ *are lower block-diagonal and* $Aut_l^0(G_\Gamma) = UT(G_\Gamma) \leftthreetimes V(\Gamma)$, *where* $UT(G_\Gamma) = Aut_l^0(G_\Gamma) \cap UT(n,R)$, *and* $UT(n,R)$ *– group of lower unitriangular matrices over* R.

Contents

Arithmeticity of $Aut_l(G_\Gamma)$

Theorem

Let R is binomial ring containt in $\mathbb{C}$. Then the group $Aut_l(G_\Gamma)$ is arithmetical when R is $\mathbb{Z}$ or field of zero characterisic.

The end