By a graph Γ we mean a finite vertex set $X = \{x_1, \ldots, x_n\}$ on which a binary adjacency relation is defined. If x_i and x_j are adjacent vertices then we say that an edge $(x_i,\, x_j)$ belongs to Γ and write $(x_i, x_j) \in \Gamma$. All graphs under consideration have no loops, i.e., $(x_i, x_i) \notin \Gamma$ for any $1 \leqslant i \leqslant n$. In correspondence with Γ is a free partially commutative group F_Γ having the following representation:

$$F_\Gamma = \langle X \mid x_i x_j = x_j x_i \;\Leftrightarrow\; (x_i,\, x_j) \in \Gamma \rangle,$$

i.e., the commutativity relation holds iff x_i and x_j are adjacent vertices.

Consider a variety $\mathcal{M}$ of groups. If to the defining relations of a partially commutative group F_Γ we add identities of the variety we obtain a partially commutative group $F_\Gamma(\mathcal{M})$ in $\mathcal{M}$ corresponding to a graph Γ. If $\mathcal{M}$ contains a variety of nilpotent groups of class two then two vertices x_i and x_j in the group $F_\Gamma(\mathcal{M})$ commute iff the two vertices are adjacent in the graph Γ. Therefore, it might be interesting to consider partially commutative metabelian groups S_Γ and partially commutative nilpotent groups $N_{c,\Gamma}$ of class $c \geqslant 2$.

In the variety $\mathbb{A}^2$ of metabelian groups, a partially commutative metabelian group is defined via the following representation:

$$S_\Gamma = \langle x_1, \ldots, x_n \mid [x_i, x_j] = 1 \Leftrightarrow (x_i,\, x_j) \in \Gamma; \mathbb{A}^2 \rangle. \qquad (*)$$

If Γ is an empty graph, then S_Γ is a free metabelian group.

Assertion 1. The commutator subgroup S' of a free metabelian group S with basis $\{x_1, \ldots, x_n\}$ is generated by commutators $[x_i, x_j]$, $1 \leqslant i < j \leqslant n$, as a $\mathbb{Z}(S/S')$-module and is torsion free.

Assertion 2. The commutator subgroup S' of a free metabelian group S with basis $\{x_1, \ldots, x_n\}$ is a free Abelian group whose basis is formed by elements $[x_i, x_j]^{p_{ijl}}$, where $1 \leqslant i < j \leqslant n$, and p_{ijl} are different monomials in variables $x_i^{\pm 1}, \ldots, x_n^{\pm 1}$.

Assertion 3. For every partially commutative metabelian group S_Γ, its abelization S_Γ / S'_Γ is a free Abelian group of finite rank n.

Assertion 4. Let $Y = \{x_{i1}, \ldots, x_{im}\}$ be a subset of a vertex set $X = \{x_1, \ldots, x_n\}$. Then a subgroup generated by the set Y in a partially commutative group S_Γ is a partially commutative metabelian group isomorphic to S_Δ, where Δ is a complete subgraph of Γ on the vertex set Y. There is a retraction $S_\Gamma \to S_\Delta$.

For any nonadjacent vertices x_i and x_j, the ideal $\mathcal{A}^{\Gamma}_{i,j}$ of a ring A is defined as follows. If x_i and x_j are in distinct connected components of the graph Γ, i.e., the distance $d(x_i, x_j)$ between these vertices is equal to infinity, then we put $\mathcal{A}^{\Gamma}_{i,j} = 0$. If $d(x_i, x_j)$ is finite then we consider all paths $\{x_i, x_{i1}, \ldots, x_{im}, x_j\}$ between x_i and x_j. With every path we associate an element $(1 - x_{i1}) \ldots (1 - x_{im})$ of the ring A. Assume $\mathcal{A}^{\Gamma}_{i,j}$ is generated by all such elements.

Definition. Let c be some element of the commutator subgroup S'_Γ. Its *annihilator* $\operatorname{Ann} c$ is an ideal in A consisting of all elements α with $c^\alpha = 1$.

THEOREM 1. Let a group S_Γ be defined via representation $(*)$ and $n \geqslant 2$. If vertices x_i and x_j are not adjacent, i.e., $[x_i, x_j] \neq 1$, then the annihilator of a commutator $[x_i, x_j]$ coincides with the ideal $\mathcal{A}_{ij}^\Gamma$.

Definition. Let M be some $A-$module. A simple ideal P of the commutative ring A called associated to the module M if exists an element $x \in A$ such that $Ann(x)$ coincides with P.

Definition. The associator $As(M)$ of a module M is the set of all associated to M ideals.

PROPOSITION 1. If for some element $1 \neq c \in S'_\Gamma$, some integer m, and some $\alpha \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we have $\alpha^m \in Ann(c)$ then $\alpha \in Ann(c)$.

THEOREM 2. If $P \in As(S_{\Gamma'})$ **then there are vertices** $\{x_{i_1}, ..., x_{i_m}\}$ **of the graph** Γ **such that the ideal** P **generated by the elements** $1 - x_{i_1}, ..., 1 - x_{i_m}$.

THEOREM 3. Let $\alpha_{ij} = \alpha_{ij}(x_i, \ldots, x_n)$**,** $i < j$**, be elements of a ring** $A = \mathbb{Z}(S_\Gamma / S'_\Gamma)$**, depending only on** $x_i, \ldots, x_n$**, and** α **be any element of** A**. Then the element**

$$c = \prod_{\{i,j \mid 1 \leqslant i < j \leqslant n,\ (x_i, x_j) \notin \Gamma\}} [x_i, x_j]^{\alpha_{ij}\alpha}$$

is equal to unity in the group S_Γ **if and only if** $\alpha_{ij}\alpha \in \mathcal{A}^{\Gamma}_{i,j}$ **for all** α_{ij}**.**

COROLLARY 1. Every element $g \in S_\Gamma$ can be written in the form

$$g = x_1^{l1} \dots x_n^{ln} \prod_{\{i,j \mid 1 \leqslant i < j \leqslant n, (x_i, x_j) \notin \Gamma\}} [x_i, x_j]^{\alpha_{ij}},$$

where $\alpha_{ij} = \alpha_{ij}(x_i, \dots, x_n)$, $i < j$, are members of the ring $A = \mathbb{Z}(S_\Gamma / S'_\Gamma)$, depending only on $x_i, \dots, x_n$, and $l1, \dots, ln$ are integers. The element g is equal to unity in S_Γ if and only if $\alpha_{ij} \in \mathcal{A}^\Gamma_{i,j}$ for all α_{ij}, and all numbers $l1, \dots, ln$ are zero.

COROLLARY 2. Let c be an element of the commutator subgroup of a partially commutative metabelian group S_Γ and

$$c = \prod_{\{i,j \mid 1 \leqslant i < j \leqslant n, (x_i, x_j) \notin \Gamma\}} [x_i, x_j]^{\alpha_{ij}}$$

be its canonical representation. An element $\alpha \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ annihilates c if and only if α annihilates all factors $[x_i, x_j]^{\alpha_{ij}}$.

THEOREM 4. Let $X = \{x_1, \ldots, x_n\}$ be a set of vertices in Γ. An element g of S_Γ lies in the centralizer $C(x_1)$ of an element x_1 in the group S_Γ if and only if

$$g = x_1^{l1} \ldots x_m^{lm} \prod_{1 \leqslant i < j \leqslant m} [x_i, x_j]^{\alpha_{ij}}, \tag{15}$$

where $x_2, \ldots, x_m$ are all vertices adjacent to x_1, $l1, \ldots, lm$ are integers, and α_{ij} are elements of $A = \mathbb{Z}(S_\Gamma / S'_\Gamma)$.

THEOREM 5. Suppose that a partially commutative metabelian group S_Γ is defined via representation $(*)$, $1 \leqslant i1 < \ldots < im \leqslant n$, assume that $q1, \ldots, qm$ are nonzero integers, and write $\mathcal{C}(g)$ for the centralizer of an element $g \in S_\Gamma$ in the commutator subgroup S'_Γ. Then

$$\mathcal{C}(x_{i1}^{q1} \ldots x_{im}^{qm}) = \mathcal{C}(x_{i1}) \cap \ldots \cap \mathcal{C}(x_{i1}).$$

THEOREM 6. Partially commutative metabelian groups S_Γ and S_Δ are elementary equivalent if and only if graphs Γ and Δ are isomorphic.

Definition. By the universal theory of a group G we mean the collection of all $\forall-$formulas true for G.

Definition. Two groups G and H are universal equivalent if their universal theories coincide.

Definition. The degree $d(x_i)$ a vertex x_i of a graph Γ is the number of the edges incidence to x_i.

Definition. We call a vertex x_i of a graph Γ the last vertex if $d(x_i) = 1$.

Notation. Let Γ be a graph. The graph Γ^* is obtained from Γ by removing all last vertices and edges incident to these vertices.

THEOREM 7. Let Γ_1 and Γ_2 be trees. The groups S_{Γ_1} and S_{Γ_2} are universal equivalent iff the graphs Γ_1^* and Γ_2^* are isomorphic.

COROLLARY 3. Let Γ_1 and Γ_2 be trees. Then the isomorphism problem is solvable for the groups S_{Γ_1} and S_{Γ_2}.