On Regularity and Differential Invariants of Geometric Structures and Lie Pseudogroups

Sarkisyan, R. A.

Moscow State University and
Financial Academy
Moscow, Russia
e-mail:
Rafael.Sarkisyan@mail.ru

This work was partially supported
by RFFI grants
№№ 05 – 01 – 01015
and 07 – 01 – 00204A

Introduction

All functions, manifolds, etc. are assumed to be real smooth.

Geometric structures are investigated in differential geometry.

Example: any tensor field on manifold **X**, such as vector field, differential form, metric, etc., is a geometric structure.

Tensor field **S** of type **(p,q)** on any manifold can be presented by function

$$S_R: U \to (T)^p \otimes (T^*)^q$$

choosing coordinate system $\mathbf{F} = (\mathbf{f}^i)$ in the neighborhood U of this manifold. If other coordinate system $\mathbf{G} = (\mathbf{g}^j)$ in **U** is chosen, **S** is represented by map

$$S(x)_G = a(S(x)_F, \partial g^j/\partial f^i).$$

Self-consistency law:

$$S(x)_H = a(S(x)_F, \partial h^j/\partial f^i)$$
$$= a(S(x)_G, \partial h^j/\partial g^i) =$$
$$a(a(S(x)_F, \partial g^j/\partial f^i), \partial h^j/\partial g^i).$$

Y-valued geometric structure

We fix arbitrary manifold **Y** and smooth self-consistent function, e.g. action

$$a: Y \times GL(\mathbf{R}) \to Y.$$

Y-valued geometric structure is represented by map $S(x)_F: U \to Y$ in coordinate system **F** and is represented by map

$$S(x)_G = a(S(x), \partial g^j/\partial f^i)$$

in another coordinate system G in U, $x \in U$.

Geometric structures of the first order: transformation law $\mathbf{a}$ depends only on first derivatives $\partial g^j/\partial f^i$.

Geometric structures of order $\mathbf{t}$: transformation law $\mathbf{a}$ depends on derivatives

$$\frac{\partial^r g^j}{(\partial f^1)^{i_1} \dots (\partial f^n)^{i_n}}$$

of order up to $\mathbf{t}$.

Example: Any connection on manifold has second order.

Formal definition:

Jets of order **q** at point 0 that keep orientation and point 0 of mappings $\mathbf{R}^n \to \mathbf{R}^n$ form Lie group $G^q(n)^0$ with respect to superpositions. Clearly, $G^1(n)^0 = GL^0_n(\mathbf{R})$.

Consider oriented manifold $X, \dim X = n$. Jet of order **q** at point 0 of map $j: Y \to X, j(0) = x$ that keeps orientation is called **q**-frame at point $x \in X$. The set of all **q**-frames of all points x of manifold X form fibre bundle $Rep_q(X) \to X$. It is a principle fibre bundle with group $G^q(n)^0$. Consider an action

$$a: G^q(n)^0 \times Y \to Y$$

on some manifold Y.

Definition: Bundle of geometric structures $P=P(Y)\to X$ of order q with typical fibre Y is the direct product $Rep_q(X)\times Y$ factorized under equivalence relation $(e,r)\sim(eg,g^{-1}r)$.

Geometric structure is section of bundle P.

Riemann metric is an important example of geometric structure.
Scalar curvature $R_S(x)$ of Riemann metric **S** is a typical example of differential invariant.
Every metric s in local coordinate system $(f^k) = F$ in the neighborhood **U** is represented by function $S_F: U\to(\mathbf{R}^*)^2$ with components

$$s_{ij}(f^k) = s_{ij}(f^k(x)),\ x\in U.$$

Then for any $x \in U$

$$R_s(x)=F(s_{ij}(x),\ \partial s_{ij}(x)/\partial f^k,\ \partial^2 s_{ij}(x)/\partial f^k \partial f^r)$$

where the value $R_s(x)$ and the form of function F (found by Riemann) are independent of coordinate system. Differential invariant of order k is such function of geometric structure components and their partial derivatives of order up to k, whose value at each point is independent of local coordinate system choosen.

Differential invariant of order k has natural domain of definition - the manifold $J^k = J^k P$ of k-jets of sections of fibre bundle of geometric structures $P=P(Y) \to X$.

Differential invariants have another definition. The (pseudo)group Diff(X) of diffeomorphisms of X acts on J^kP in a naturally way.

Differential invariant of order k may be defined as function on J^kP, which is locally constant on each orbit of this action.

Definition. The action of a group or a pseudogroup on some manifold is said to be regular at a point z of this manifold if the dimension of orbits of this action is constant in some neighborhood of z.

It is easy to show that regular points form open dense subset in J^kP for each k.

Theorem 1. Let $P=P(D)\to X$ be a fibre bundle of geometric structure with typical fiber D and differential order q. Let dimD=m, dimX=n and m>n. Then at any regular point $a\in J^kP$ there are at least

$$t(k) = mC_{n+k}^{k} - n(C_{n+k+q}^{k+q} - 1)$$

functionally independent differential invariants of order k, defined at a.

Theorem 1 continues…

For $k \to \infty$ we have

$$t(k) = (m - n) C_{n+k}^{k} - e(k) C_{n+k}^{k}$$

with $e(k) \to 0$, where $e(k)$ depends only on m,n,q and k. Therefore, $t(k) \to \infty$ as $k \to \infty$.

Note that exact values of $t(k)$ for general Riemann metrics and some other specific geometric structures are computed in [Thomas 1934].

A bundle of geometric structures $E \to X$ is called special bundle if the dimension m of its fiber is smaller than the dimension n of it's base X; otherwise, E is called non-special bundle.

Theorem 1 deals with non-special bundles.

Now we will consider special bundles. All results listed below are valid for the case $n > 2$. (One may deal with the cases $n = 1$ and $n = 2$ in the similar way by changing some formulations of the results.)

1. Each special manifold E at each regular point x (i.e. in some neighborhood of x) is locally isomorphic to one of the 19 types of sample manifolds E_i.

This local isomorphism is natural, i.e. commutes with action of Diff(X) on E and E_i.

2. For all sample manifolds, the action of the group Diff(X) on E_i is described.

3. If a local sections s of special bundle is sufficiently general at point $b \in X$, than s may be reduced to canonical form in a neighborhood of b. All this forms are listed.

4. For any special bundle E the finite complete set T of functionally independent differential invariants is
written out explicitly.

Completeness of set T means that each differential invariant (of arbitrary differential degree) at any sufficiently general point of E may be represented as a superposition of invariants, belonging to set T.

Now we will list types of sample manifolds. Each sample manifold E_i is defined by an action

$$G^q(n)^0 \times Y_i \to Y_i$$

on some manifold Y_i (Y_i is a typical fiber of E_i). There are 17 types of sample manifolds corresponding to q=1, i.e. having first order. For q=1 we have $G^q(n)^0 = GL^0_n(\mathbf{R})$. All following constructions are local. Let $x^1,\dots,x^n$ be a coordinate system on X. Fix a point $b \in X$. We name sufficiently general in b section shortly by b-section.

Type 1. Corresponds to trivial action of $GL^0_n(\mathbf{R})$ on $\mathbf{R}^m (m \leq n)$ and defines trivial bundle $E_1 = \mathbf{R}^m \times X$ on X. A b-section $s: X \to E_1$ is given by m functions $s^1, \ldots, s^m$ such that Jacoby matrix $(\partial s/\partial x)(b)$ has rank m. If $m<n$, canonical form of s (in corresponding coordinate system $f^1, \ldots, f^n$ on X) is $s^1 = f^1 + \lambda^1, \ldots, s^m = f^m + \lambda^m$. If $m=n$, then we also have another canonical form

$$s^i = f^i + \lambda^i \text{ for } i<n, \; s^n = -f^n + \lambda^n$$

(λ^i are constants).

Type 2. $E_2=(\mathbf{R}^{m-1}\times X)_X+\Lambda^{n+}$ is a direct sum of trivial bundle $\mathbf{R}^{m-1}\times X$ and one-dimensional bundle Λ^{n+} of positive n-forms on X. A b-section $s:X\rightarrow E_2$ is given by m-1 functions $s^1,\dots,s^{m-1}$ such that Jacoby matrix $(\partial s/\partial x)(b)$ has rank m-1, and any positive n-form π. Canonical form of s is $s^1=f^1+\lambda^1,\dots,s^{m-1}=f^{m-1}+\lambda^{m-1}$, $\pi=df^1\wedge\dots\wedge df^n$.

Type 6. E_6 is the positive projectivezation P^+TX (this means that we identify vectors e and λe for positive λ) of the tangent space of X. Any section s has as canonical representative vector field $\partial/\partial f^1$ on X (rectifyability of vector field).

Type 3. $E_3=(\mathbf{R}\times X)_X+P^+TX$ is a direct sum of one-dimensional trivial bundle and P^+TX. A b-section $s=(\Psi, \bar{e})$ is given by condition $\partial\Psi/\partial e(b)\neq 0$. Canonical forms are

$$\Psi= \lambda^1\pm f^1, e= \partial/\partial f^1.$$

For further purposes for any 1-form ω on X and any natural $k \leq n$ denote by $\omega^{(k)}$ the k-form $\underbrace{d\omega\wedge\ldots\wedge d\omega}_{a\ times}$ if k=2a and $\underbrace{d\omega\wedge\ldots\wedge d\omega}_{a\ times}\wedge\omega$ if k=2a+1.

Type 7(n=2a). E_7 is the positive projectivezation $P^+\Omega X$ of the cotangent space of X. Any b-section s is represented by 1-form ω on X with the condition $\omega^{(n-1)}(b)\neq 0$. Using freedom in choosing ω and Darboux theorem, in convenient coorinate system h^i on X we have

$$\omega = \Sigma_{i=1}^{a} h^{2i-1} dh^{2i},\ h^1(b)\neq 0,\ h^i(b)=0 \text{ for } i>1.$$

In coordinate system:

$f^1 = h^1 - h^1(b)$, $f^{2i-1} = h^{2i-1}/h^1$ for $2\leq i\leq a$, $h^{2i} = f^{2i}$ for $1\leq i\leq a$ section s has to canonical representative

$$\omega = df^2 + \Sigma_{i=2}^{a} f^{2i-1} df^{2i}.$$

Type 8(n=2a+1). E_8 is the positive projectivezation $P^+\Omega X$ of the cotangent space of X. Any b-section s is represented by 1-form ω on X with the condition $\omega^{(n)}(b)\neq 0$. Canonical forms are:

$$\omega=\Sigma_{i=1}^{a} f^{2i-1}df^{2i}\pm df^{2a+1}.$$

Type 4(n=2a). $E_4 = (\mathbf{R}\times X)_X + P^+\Omega X$ is a direct sum of one-dimensional trivial bundle and $P^+\Omega X$. A b-section $s=(\Psi, \bar{\omega})$ is given by condition $d\Psi\wedge\omega^{(n-1)}\neq 0$. Canonical forms are

$$\Psi=\lambda^1\pm f^1,\ \omega= df^2 +\Sigma_{i=2}^{a} f^{2i-1}df^{2i}.$$

Type 5(n=2a+1). $E_5 = (\mathbf{R}\times X)_X + P^+\Omega X$ is a direct sum of one-dimensional trivial bundle and $P^+\Omega X$. A b-section $s=(\Psi, \bar{\omega})$ is given by conditions

$$d\Psi\wedge\omega^{(n)}(b)\neq 0, (d\omega^{(2a-1)}\wedge d\Psi)(b) \neq 0.$$

Canonical forms are

$$\Psi=\lambda^2+f^2,$$

$$\omega=\Sigma_{i=1}^{a} f^{2i-1}df^{2i}\pm df^{2a+1}.$$

Type 9. $E_9=P^+TX+\Lambda^{n+}$. A b-section is a pair (e, π) such that $e(b)\neq 0$. Canonical form is $e=\partial/\partial f^n$, $\pi=df^1\wedge\ldots\wedge df^n$.

Type 10(n=2a). $E_{10} = P^+\Omega X + \Lambda^{n+}$. A b-section is a pair (ω,π) such that $\omega^{(n-1)}(b)\neq 0$. Canonical form is $\omega=df^2+\Sigma_{i=2}^{a} f^{2i-1}df^{2i}$, $\pi= df^1\wedge\ldots\wedge df^n$.

Type 11(n=2a+1). $E_{11} = P^{+}\Omega X + \Lambda^{n+}$. A b-section is a pair (ω,π) such that $\omega^{(n)}(b)\neq 0$. Canonical forms are $\omega=\Sigma_{i=1}{}^{a} f^{2i-1}df^{2i} \pm df^{2a+1}$, $\pi=df^{1}\wedge\ldots\wedge df^{n}$.

Type 12. In this case Y_{12} (typical fiber of E_{12}) is equal to $\mathbf{R}^{n} - 0$, and $g\in GL^{0}{}_{n}(\mathbf{R})$ acts on $v\in\mathbf{R}^{n}$ by formula $gv=\det(g)^{(d-1)/n}gv$. Here $d\in\mathbf{R}$ is a parameter, and we call E_{12} quasitangent bundle and denote $E_{12}=T_{(d)}X$ ($E_{12} = TX$ when d=1). For each coordinate system in X we identify (non-canonically) sections of $T_{(d)}X$ with vector fields on X. When $d \neq 1-n$ then a b-section e of $T_{(d)}X$ is defined by condition $e(b)\neq 0$ and has canonical form $e= \partial/\partial f^{1}$.

Case d = 1-n. There is a coordinate system h^i in X such that $e^1>0$, $e^i=0$ for i>1. A b-section is defined by condition $[\partial e^1/\partial h^1](b)\neq 0$ and has canonical forms $e^1= 1\pm f^1$, $e^i=0$ for i>1.

Types 13 and 14. In this case $Y_{13,14}$ (typical fiber of $E_{13,14}$) is equal to $\mathbf{R}^{*n} - 0$, and $g\in GL^0{}_n(\mathbf{R})$ acts on $v\in\mathbf{R}^{*n}$ by formula

$$gv=\det(g)^{(d-1)/n}(g^{-1})^T v.$$

Here $d\in\mathbf{R}$ is a parameter, and we call $E_{13,14}$ quasicotangent bundle and denote

$$E_{13,14}=\Omega_{(d)}X \quad (E_{13,14} = \Omega X \text{ when } d=1).$$

Case d=1 is well known so we assume $d\neq 1$. For each coordinate system in X we identify (non-canonically) sections of $T_{(d)}X$ with 1-forms on X.

Type 13(n=2a). A b-section ω is defined by condition $\omega^{(n-1)}(b)\neq 0$ and has canonical form $(1+f^1)df^2 + \Sigma_{i=2}{}^a f^{2i-1}df^{2i}$.

Type 14(n=2a+1). When $(d-1)(a+1)\neq n$ then a b-section ω is defined by condition $\omega^{(n)}(b)\neq 0$ and has canonical forms $\omega = \Sigma_{i=1}{}^a f^{2i-1}df^{2i} \pm df^{2a+1}$. In the case $(d-1)(a+1)=n$ the definition of b-section is a little too long and we omit it; canonical forms are

$$\omega = (f^2+\lambda^2)(\Sigma_{i=1}{}^a f^{2i-1}df^{2i}\pm df^{2a+1}),\ f^i(b)=0.$$

Type 18(n=4). E_{18} is a grassman manifold E_{gr} of oriented 2-planes of $\mathbf{R}^4$.

The definition of b-ection is a little too long and we omit it again. Any b-section s of E_{gr} has canonical form: s is generated by vector fields $e^U = \partial/\partial f^4$, $d^U = f^4\partial/\partial f^1 + f^3 f^4 \partial/\partial f^2 + \partial/\partial f^3$.

Type 19(n=3). E_{19} is a manifold consisting of oriented flags s=(l,p) in $\mathbf{R}^3$, namely each flag consists of oriented line l, contained in oriented 2-plane p. We denote E_{19} by E_{fl}. To define b-section we note that any section s(x)=(l(x),p(x)) of E_{fl} defines (nonuniquely) 1-form $\omega(x)$ on X (p(x) annihilates $\omega(x)$), and condition $[\omega \wedge d\omega](b) \neq 0$ defines b-section. The canonical forms for b-section are:

$$l(x)=\{\partial/\partial f^1\},\ p(x)=\{\partial/\partial f^1,\ \partial/\partial f^2 \pm f^1\partial/\partial f^3\},$$
$$f^i(b)=0.$$

Now we will list 3 types of sample manifolds for $q=2$.

The group $G^2(n)^0$ has a natural projection on Aff_n (Aff_n is the affine group of $\mathbf{R*}^n$) and $G^2(n)^0$ acts on $Y_{15,16,17}$ via this projection.

$Y_{16} = \mathbf{R*}^n$ ($n=2a$), $Y_{17} = \mathbf{R*}^n$ ($n=2a+1$), and Aff_n acts on $\mathbf{R*}^n$ in a standard way. Y_{15} is a manifold of all hyperplanes (not necessarily containing null) in $\mathbf{R*}^n$. We identify a section s of E_i (non-canonically) for $i=16,17$ with a 1-form $\omega \in \Omega X$, and for $i=15$ with a family of hyperplanes in ΩX.

Type 15. Any local section s of E_{15} is a b-section. The canonical form of s is given by equation $\partial/\partial f^1 = 0$ ($\partial/\partial f^i$ and f^i form a coordinate system in ΩX).

Type 16. A b-section of E_{16} is given by condition $\omega^{(n)}(b)\neq 0$. Canonical forms are

$$\omega = \pm f^1 df^2 + \Sigma_{i=2}^{a} f^{2i-1}df^{2i}.$$

Type 17. A b-section of E_{17} is given by condition $\omega^{(n-1)}(b)\neq 0$. Canonical form is

$$\omega = \Sigma_{i=1}^{a} f^{2i-1}df^{2i} + df^{2a+1}.$$

There are no special bundles for $q\geq 3$.

List once more all types of sample manifolds.

q=1.

$$E_1=\mathbf{R}^m\times X, \quad E_2=(\mathbf{R}^{m-1}\times X)_X+\Lambda^{n+},$$
$$E_3=(\mathbf{R}\times X)_X+P^+TX, \quad E_{4,5}=(\mathbf{R}\times X)_X+P^+\Omega X,$$
$$E_6=P^+TX, \quad E_{7,8}=P^+\Omega X, \quad E_9=P^+TX_X+\Lambda^{n+},$$
$$E_{10,11}=P^+\Omega X_X+\Lambda^{n+}, \quad E_{12}=T_{(d)}X, \quad E_{13,14}=\Omega_{(d)}X,$$
$$E_{18}=E_{gr}, \quad E_{19}=E_{fl}.$$

q=2. E_{15}, E_{16}, E_{17}.

Further discussion is valid for an arbitrary Lie pseudogroup PG of order q acting on arbitrary manifold P of dimension m + n and not only for bundles of geometric structures. Let us consider the spaces $J^k P$ and $J^\infty P$ consisting respectively of jets of order k and infinite jets of submanifolds of dimension n in the manifold P. For $k \geq s$, there is the natural projection $\pi(s,k)$: $J^s P \rightarrow J^k P$. The space $J^\infty P$ has an ordinary weak topology of projecttive limit. The action of the pseudo-group PG may be naturally extended to all manifolds $J^k P$ and to the space $J^\infty P$.

Then, we have:

(1) P has an open everywhere dense invariant set R_0 consisting of regular points.

(2) For each natural number k, there is an open everywhere dense invariant set R_k in J^kP consisting of regular points.

(3) There is some natural number $K = K(n,m,q)$, depending only of n, m and q such that for each $k \geq K$ we have $\pi^{-1}(k,K)[R_K] \subseteq R_k$, i.e. each preimage of each point from R_K is regular. Let us define $R^\infty = \pi^{-1}(k,K)[R_K]$. This set is open everywhere dense invariant domain in $J^\infty P$.

(4) If the number of functionally independent differential invariants at point $z \in R^{\infty}$ is not finite, there is a finite set (depending of z) of differential invariants $F_1, \ldots F_s$ and n invariant differentiations $D_1, \ldots, D_n$, such that acting repeatedly by D_i on F_j one gets complete set of differential invariants for each point close to point z (Tresse's Theorem). In a sense, for each point $z \in R^{\infty}$ functions F_j and invariant differentiations D_i may be explicitly found (using differentiations, algebraic operations and solution of systems of algebraic equations via implicit function theorem). The number $P_z(k)$ of functionally independent differential invariants at point z having differential degree k is a polynomial in k (for sufficiently large k). We call this polynomial the Hilbert's polynomial of the point z.

(5) The domain R^{∞} is a disjoint union of finite number of open sets (atoms), every point of each atom having the same Hilbert polynomial.

In principle one can explicitly derive conditions defining each atom from Lie pseudogroup PG equations. In principle for each atom the corresponding Hilbert polynomial can be computed explicitly. Hilbert polynomials for all atoms are contained in the universal list of Hilbert polynomials. This list is finite and depends only on m, n and q, and does not depend on given pseudogroup PG.

In principle one can explicitly find this list. (Explicitly means that one uses only finite number of differentiations and algebraic operations.)

All results are true both in smooth and real analytic cases.

In real analytic case domain R^∞ is a single atom and it's complement in J^∞ is contained in a hypersurface.

Many authors considered differential invariants and Tresse's theorem. We mention both classical works of Lie,Cartan and Tresse, and modern works of Thomas T.Y., Ovsiannikov, Kumpera, Olver, Bryant, Ibragimov, Kruglikov & Lychagin, Yumaguzhin, Morozov among others.

As pointed out in Arnold's paper "Mathematical problems in classical physics" in Trends and Perspectives in Applied Mathematics (Appl. Math. Series, vol. 100), Springer, 1994, pp. 1-20], and in book"Arnold's Problems", Moscow: FAZIS, 2000; problem 1994-24, this theorem was formulated by Tresse, but it's proof has not yet been absorbed by modern mathematics.

A. Kumpera in the paper "Invariants differrentiels d'un pseudo-groupe de Lie", J. differential geometry, v.10(1975), p.p. 289-416 has found sufficient conditions for validity of Tresse's theorem. Author has checked the validity of this conditions for the points of the domain R^{∞}.

This work uses some ideas, which was communicated to author by A. Khovanskii about 20 years ago. Now they are published in the book A. Khovanskii, S. Chulkov, "The geometry of semigroup $Z^n_{\leq 0}$", M., 2006.

The results reported are published in papers:

1. R. Sarkisyan, On differential invariants of geometric structures, Izvestiya: Mathematics 70:2(2006), 307–362.

2. R. Sarkisyan, On differential invariants of geometric structures II, submitted for publication.

3. R. Sarkisyan and I. Shandra, Regularity and Tresse’s theorem for geometric structures, Izvestiya: Mathematics 72:2(2008), 345–382.

4. R. Sarkisyan, Rationality of Poincare series in local problems of Analisys according to Arnold, Izvestiya: Mathematics 74:2(2010).