Algebraic Geometry over Solvable Groups

Nikolay Romanovskiy

"New Algebraic-Logical Methods in Solution for Systems of Equations in Algebraic Structures" August, 16-22, 2009, Omsk, Russia

Algebraic geometry over groups and other systems.
G.Baumslag, A.Myasnikov, B.Plotkin and V.Remeslennikov.

Basic papers:

G.Baumslag, A.Myasnikov and V.Remeslennikov, Algebraic geometry over groups. I. Algebraic sets and ideal theory, J. Algebra, 219, N 1 (1999), 16-79.
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Our talk based on following papers:

C.K.Gupta, N.S.Romanovskiy, *The property of being equationally Noetherian for some soluble groups*, Algebra and Logic, 46(1), 2007, pp. 28-36.
A.Myasnikov, N.Romanovskiy, *Krull dimension of solvable groups*, arXiv:0808.2932v1 [math.GR] (http://xxx.lanl.gov:80/abs/0808.2932v1).
N.S.Romanovskiy, *Divisible rigid groups*, Algebra and Logic, 47(6), 2008, pp. 426-434.
N.S.Romanovskiy, *Equational Noetherianess of rigid solvable groups*, Algebra and Logic, 48(2), 2009, 147-160.
N.S.Romanovskiy, *Irreducible algebraic sets over divisible decomposed rigid groups*, submitted for publication.
A.Myasnikov, N.S.Romanovskiy, *On universal theories of rigid solvable groups*, submitted for publication.

A-group, A-subgroups, A-homomorphism, ...

$F = A * \langle x_1, \ldots, x_n \rangle,\ x = (x_1, \ldots, x_n)$.
Equation $v(x) = 1,\ v(x) \in F$.
F is a group of equations.
$\{v_i(x) = 1\ (i \in I)\}$, set of solutions $S \subseteq A^n$ is algebraic set.

Annulator of S: $I(S) = \{v(x) \in F \mid v(S) = 1\}$.
Coordinate group of S: $\Gamma(S) = F/I(S)$.
$\Gamma(S) \geqslant A$.

A category of algebraic sets is equivalent to a category of coordinate groups.

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Group of equations $D = \langle A, x_1, \ldots, x_n \rangle$:
$x \to (a_1, \ldots, a_n) \in A^n$ possible to continue to A-epimorphism $D \to A$.

$D = F/H$. H is maximal $= I(A^n)$ is a set of all A-identities, $F/H = \Gamma(A^n)$.

In particular, if $A \in \mathfrak{M}$, then $\Gamma(A^n) \in \mathfrak{M}$.

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The intersection of algebraic sets is algebraic, but the union is not in general case.

Zariski topology on A^n: we take the algebraic sets as a sub-basis for the closed sets.

The topology is Noetherian, if it satisfies the descending chain condition on closed sets. In this case $S = S_1 \cup \ldots \cup S_k$, where S_i are irreducible algebraic sets.

We say that given group is equationally Noetherian if for any n arbitrary system of equations on $x_1, \ldots, x_n$ over this group is equivalent to some finite subsystem.
A group A is equationally Noetherian if and only if for any n Zariski topology on A^n is Noetherian.

Hard to study algebraic geometry over given group without last property. So, to be equationally Noetherian group is necessary condition for good algebraic geometry.

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1) Abelian groups.

2) Linear groups, in particular, free groups.

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Example 1.
$A = \langle a_1, a_2, \ldots, b_1, b_2, \ldots \ |\mathfrak{N}_2,\ [b_1, a_1] = 1,$
$[b_2, a_1] = [b_2, a_2] = 1, \ldots, [b_n, a_1] = \ldots = [b_n, a_n] = 1, \ldots \rangle$
A system of equations $\{[x, a_i] = 1\}$ isn't equivalent to a finite subsystem.

Example 2.
$H = \langle c, d \rangle$ is a free centre-by-metabelian group, $[H, H]$ contains a free nilpotent group of class 2 with a countable basis $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$.
$G = \langle c, d \mid \ldots \rangle \geqslant A.$

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Theorem 1 (C.K.Gupta, N.Romanovskiy, 2007)

Free solvable and close to them groups are equationally Noetherian.

Classes of groups where algebraic geometry was developing:

Free groups (Myasnikov, Kharlampovich, Sela, Remeslennikov, Chiswell ...).

Finitely generated nilpotent groups (Remeslennikov, Amaglobeli, Mishenko).

Metabeliah groups which are close to free metabelian groups (Remeslennikov, Stohr, Chapuis, Romanovskiy, Timoshenko).

Solvable groups of class ≥ 3 which are close to free solvable groups (Romanovskiy, Myasnikov, K.Gupta).

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Definition

p-rigid group G: there is a normal series

$$G = G_1 > G_2 > \ldots > G_p > G_{p+1} = 1,$$

G_i/G_{i+1} are abelian and considering as right $\mathbb{Z}[G/G_i]$-modules has no torsion.

Why rigid? - this series is unique and given group G is solvable of class p (Myasnikov, Romanovskiy).

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In our Theorem 1 "close to free soluble groups"means finitely generated rigid groups.

Finitely generated rigid groups are exactly finitely generated subgroups of iterated wreath products of free abelian groups $A_n \wr (A_{n-1} \wr (\ldots \wr A_1)\ldots)$.

$T_i = G_i/G_{i+1}$ is a torsion free module over the ring $\mathbb{Z}[G/G_i]$.
$r_i(G) = \mathrm{rank}\ T_i,\ r(G) = (r_1(G), \ldots, r_p(G))$.
$\mathbb{Z}[G/G_i]$ is an Ore ring and embeds into the skew field of fractions $Q(G/G_i)$.
T_i embeds into the right vector space
$V_i = T_i \otimes_{\mathbb{Z}[G/G_i]} Q(G/G_i)$, $r_i(G) = \dim V_i$.

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Theorem 2 (Myasnikov, Romanovskiy)

Let A be a finitely generated p-rigid group, $S \subseteq A^n$ be an irreducible algebraic set, $G = \Gamma(S)$.
1) Then G is p-rigid.
2) Let G_i and A_i denote corresponding terms of rigid series of groups G and A. If some system elements of A_i/A_{i+1} is linear independent over the ring $\mathbb{Z}[A/A_i]$, then it is linear independent over the ring $\mathbb{Z}[G/G_i]$. So, we can define codimension G_i/G_{i+1} over A_i/A_{i+1} which denote by $d_i(S)$.
3) Inequality $d_i(S) \leq n$ holds.

Let $d(S) = (d_1(S), \ldots, d_p(S))$. In the theorem 2 all ranks $r_i(G)$ are finite, so $d(S) = r(G) - r(A)$.

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For property 2) in the theorem we say that A is embedded into G with preserving linear independence.

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Let A be a finitely generated rigid group, S and P irreducible algebraic subsets of A^n. If $S > P$, then $d(S) > d(P)$ in lexicographic order.

Remind that topological dimension of given topological space by definition is equal to supremum of lengths of chains $S_1 > S_2 > \ldots > S_m$ irreducible subsets.

Corollary

If A be a p-rigid group then the topological dimension of the space A^n is finite and doesn't exceed the number $(n+1)^p$.

For free group F we don't know: is the topological dimension of F^n finite or not?

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V.N.Remeslennikov, N.S.Romanovskiy, Irreducible algebraic sets in metabelian groups, Algebra and Logic, 44(5), 2005, pp. 336-347, N.S.Romanovskiy, Algebraic sets in metabelian groups, Algebra and Logic, 46(4), 2005, pp. 503-513
we described algebraic sets in the dimension 1 over free metabelian group.
This description doesn't give any optimism that possible to get good information about all algebraic sets over arbitrary (finitely generated) rigid group.

To find such class of p-rigid groups, that any p-rigid group can be embedded into some group of this class and algebraic geometry over groups of the class will be "good".

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Definition

Rigid group G is called divisible if any factor $T_i = G_i/G_{i+1}$ is a divisible module over the ring $\mathbb{Z}[G/G_i]$ or, in other words, T_i is a vector space over skew field of fractions $Q(G/G_i)$.

Let $\alpha_1, \dots, \alpha_p$ be nonzero cardinalities. Construct group $M(\alpha_1, \dots, \alpha_p)$ by induction. $M(\alpha_1)$ is a direct sum of α_1 copies of $\mathbb{Q}$. Let $A = M(\alpha_1, \dots, \alpha_{p-1})$ and T be a vector space with a basis of cardinality α_p over the skew field $Q(A)$. Then set

$$M(\alpha_1, \dots, \alpha_p) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

We call such group decomposed divisible rigid group.

Theorem 4

Arbitrary p-rigid group can be embedded into some decomposed divisible p-rigid group.

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Definition

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Let $\alpha_1, \ldots, \alpha_p$ be nonzero cardinalities. Construct group $M(\alpha_1, \ldots, \alpha_p)$ by induction. $M(\alpha_1)$ is a direct sum of α_1 copies of $\mathbb{Q}$. Let $A = M(\alpha_1, \ldots, \alpha_{p-1})$ and T be a vector space with a basis of cardinality α_p over the skew field $Q(A)$. Then set

$$M(\alpha_1, \ldots, \alpha_p) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

We call such group decomposed divisible rigid group.

Theorem 4

Arbitrary p-rigid group can be embedded into some decomposed divisible p-rigid group.

Theorem 5

Let A be a p-rigid subgroup of divisible rigid group H. Then there is a minimal divisible subgroup containing A, let it be G = divisible closure A in H. This subgroup G is p-rigid and G_i/G_{i+1} is generated by the set A_i/A_{i+1} as a vector space over $Q(G/G_i)$, in particular, $r_i(A) \geq r_i(G)$.

Theorem 6

For given p-rigid group A there is such divisible closure G that A is embedded into G with preserving linear independence. We call G divisible completion of A. Any two divisible completions of A are A-isomorphic.

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For developing algebraic geometry over divisible rigid groups we need to prove that such groups are equationally Noetherian (we know only from the theorem 1 that finitely generated rigid groups are equationally Noetherian).

Theorem 7

Any rigid group is equationally Noetherian.

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Theorem 8

Let $M = M(\alpha_1, \ldots, \alpha_p)$. Then finitely generated M-group G is a coordinate group of some irreducible algebraic set over M if and only if G is p-rigid and M is embedded into G with preserving linear independence.

We deduce the theorem 8 from following statement.

Theorem 9

Let a group G contain $M = M(\alpha_1, \ldots, \alpha_p)$ as a subgroup. Then G is M-universally equivalent to M if and only if G is p-rigid and M is embedded into G with preserving linear independence.

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Theorem 2' (Myasnikov, Romanovskiy)

Let A be a p-rigid group, $S \subseteq A^n$ be an irreducible algebraic set, $G = \Gamma(S)$.
1) *Then G is p-rigid.*
2) *Let G_i and A_i denote corresponding terms of rigid series of groups G and A. If some system elements of A_i/A_{i+1} is linear independent over the ring $\mathbb{Z}[A/A_i]$, then it is linear independent over the ring $\mathbb{Z}[G/G_i]$. So, we can define codimension G_i/G_{i+1} over A_i/A_{i+1} which denote by $d_i(S)$.*
3) *Inequality $d_i(S) \leq n$ holds.*

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Malcev proved that free solvable group of class ≥ 2 has unsolvable elementary theory. The universal theory of free metabelian group was studied by Chapuis, Remeslennikov and Stohr, in particular, it is solvable. Free solvable groups of given class ≥ 3 and different ranks are universally equivalent too and their universal theory is unsolvable if the universal theory of rational numbers is unsolvable (Chapuis). Nevertheless, Chapuis proved that the the universal theory of an iterated wreath product of of free abelian groups is solvable.

We construct a recursive system of axioms which define p-rigid groups in the class of all p-soluble groups.

Let F denote a free solvable group of class p, G denote an arbitrary p-rigid group, W denote the iterated wreath product of of p cyclic groups. For corresponding universal theories we prove conclusions

$$\mathcal{A}(F) \supseteq \mathcal{A}(G) \supseteq \mathcal{A}(W).$$

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Then we study the universal theory of the group $M = M(\alpha_1, \ldots, \alpha_p)$ with constants from M and describe M-groups which are M-universally equivalent to M. Using this description we prove

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