Limit Algebras

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Motivation

Let $\mathcal{A}$ be an algebraic structure (an algebra) in a functional language $\mathcal{L}$ (with no predicates).

Slogan

To have a good understanding of $\mathcal{A}$ it is necessary to investigate a structure of finitely generated algebras from the universal close **Ucl**($\mathcal{A}$) of $\mathcal{A}$.

Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{C})$, *i.e.*, $\mathcal{C} \in$ **Ucl**$(\mathcal{A})$;
2. $\mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.

The problem solved by Emil Artin

Let $f(x_1, \ldots, x_n)$ be a rational function over the real number field $\mathbb{R}$. If $f(x_1, \ldots, x_n) \geqslant 0$ for all $(x_1, \ldots, x_n)$ in the domain of f then the function $f(x_1, \ldots, x_n)$ may be represented as a sum of squares of rational functions over $\mathbb{R}$.

The attempt to solve this problem has made Artin and Schreier to introduce the notion of linear ordered fields and to work not only with $\mathbb{R}$, but also with linear ordered fields, in where universal theory contains the universal theory of $\mathbb{R}$.

Alfred Tarski asked

Problem 1: whether the free groups on two or more generators have the same first order theory,

Problem 2: and whether elementary theory of concrete free group is decidable.

The first attempt to give definition of limit group

Let F be a free nonabelian group and G finitely generated group. Then

$$G \in \mathbf{Ucl}(F) \quad \Longleftrightarrow \quad G \text{ is a fully residually free group.}$$

Let $\mathbf{K}_{\mathrm{fg}}$ be a class of all finite groups, and $\mathrm{Th}_{\forall}(\mathbf{K}_{\mathrm{fg}})$ universal theory of the class $\mathbf{K}_{\mathrm{fg}}$.

Theorem (Budkin, Gorbunov, 1975)

Finitely presented limit groups for $\mathbf{K}_{\mathrm{fg}}$ = Finitely presented residually finite groups.

Corollary

Universal theory of the class of all finite groups is more rich than universal theory of the class of all groups.

Let $\mathbf{K}_{\mathrm{ff}}$ be a class of all finite fields, and
$T_{\mathrm{ff}} = \mathrm{Th}_{\forall}(\mathbf{K}_{\mathrm{ff}})$ universal theory of the class $\mathbf{K}_{\mathrm{ff}}$.

Definition

Fields, which are models of the theory T_{ff}, are called quasifinite fields.

Theorem (Ax-Kochen, Ershov)

Universal theory of the class of all finite fields is more rich than universal theory of the class of all fields.

Fact

$$\mathrm{Th}_\forall(F_r) = \mathrm{Th}_\forall(F_s), \quad r, s \geqslant 2.$$

So, we may talk about the universal theory $T_{\text{free gr}} = \mathrm{Th}_\forall(F)$ of non-abelian free group F.

Theorem (Remeslennikov, 1989)

Finitely generated group G is a model of the theory $T_{\text{free gr}}$ iff it is fully residually free.

- V. N. Remeslennikov, $\exists$-*free groups*, Sib. Math. J., **30 (6)**, 1989, pp. 998–1001.

Let F be free nonabelian group and G a finitely generated group.

Definition

A sequence of homomorphisms $f = (f_i : G \to F)$ is termed convergent (stable), if for any $g \in G$ elements $f_i(g)$ is eventually either always trivial or always non-trivial.

To a convergent sequences is associated the $\ker f = \varinjlim f_i$ of elements that are eventually trivial.

The First Definition of Limit Group for Free Group

An (algebraic) limit group (over F) is any group of the form

$$\Gamma = G/\ker f$$

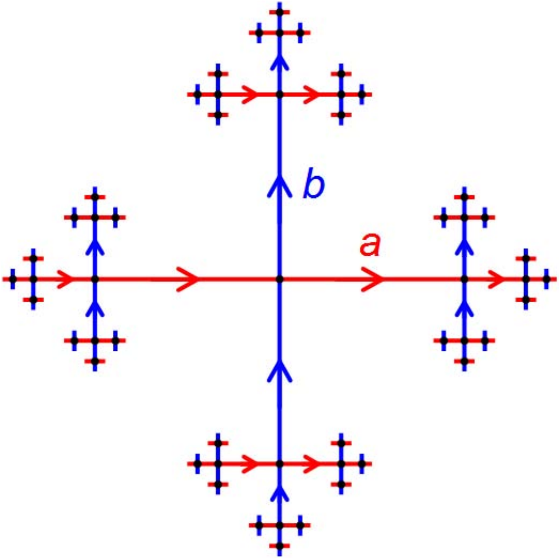
for $f = (f_i)$ a stable sequence of homomorphisms $f_i : G \to F$.

Example

A fully residually free group $\longleftrightarrow$ An algebraic limit group.

Cayley Graph for the free group of rank 2

$F_2 = \langle a, b \rangle$



Graph $\mathrm{Cal}(F_2)$ is a simplicial (combinatory) tree, and group F_2 acts on $\mathrm{Cal}(F_2)$ by right multiplication with no fixed-points.

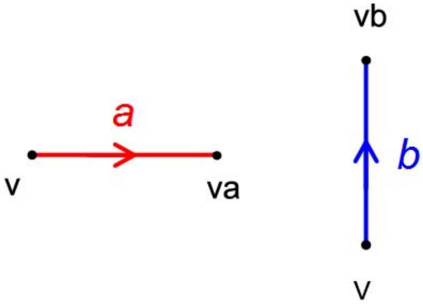


Figure: Acting by elements *a* and *b*.

The Second Definition (Geometric) of Limit Group for Free Group

Let $f = (f_i : G \to F)$ be a stable sequence of homomorphisms. Then G acts on Cayley graph T of F by the following way:

$$g \in G, v \in T \quad \longrightarrow \quad g(v) = f_i(g) \cdot v.$$

Details of this definition see in papers:

- H. J. R. Wilton, *An Introduction to Limit Groups*, Preprint, 2005.
- G. Champetier, V. Guirardel, *Limit groups as limits of free groups*, Israel J. Math., **146**, 2005, pp. 1–75.

Theorem (Kharlampovich, Miasnikov, 1998)

A finitely generated group is a limit group for free group iff it is a subgroup of iterated extensions of centralizers of a free group.

Definition

Let G be a group and $C(g)$ centralizer of some element $g \in G$. Then we say that group

$$G_C(g) = \langle G, t \mid \mathrm{rel}(G), th = ht, h \in C(g) \rangle$$

is extension of centralizer of G.

Another description of limit groups in terms of generalized doubles see in Sela's preprint, 2001.

- O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free group I: Irreducibility of quadratic equations and Nullstellensatz*, J. Algebra, **200 (2)**, 1998, pp. 472–516.
- O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free group II: Systems in trangular quasi-quadratic form and description of residually free groups*, J. Algebra, **200(2)**, 1998, pp. 517–570.
- V. N. Remeslennikov, $\exists$*-free groups*, Sib. Math. J., **30 (6)**, 1989, pp. 998–1001.

Let $X = \{x_1, \ldots, x_n\}$ be a finite set of variables and T is a theory.

Definition

A set p of atomic or negations of atomic formulas in variables X is called an atomic type relative to a theory T if $p \cup T$ is consistent. A maximal atomic type in variables X with respect to inclusion is termed a complete atomic type of T.

If p is a complete atomic type in variables X then for every atomic formula φ in variables X either $\varphi \in p$ or $\neg \varphi \in p$.

Example

Let $\mathcal{A}$ be an algebra in a language $\mathcal{L}$ and $\bar{a} = (a_1, \ldots, a_n) \in A^n$. Then the set $\mathrm{atp}^{\mathcal{A}}(\bar{a})$ of atomic or negations of atomic formulas in variables X that are true in $\mathcal{A}$ under an interpretation $x_i \mapsto a_i$, $i = 1, \ldots, n$, is a complete atomic type relative to any theory T such that $\mathcal{A} \in \mathrm{Mod}(T)$.

Every complete atomic type p in variables X defines congruence $\sim_p$ on the free algebra of terms in the language $\mathcal{L}$:

$$t \sim_p s \iff (t = s) \in p, \quad t, s \text{ are terms.}$$

Definition

We say that factor-algebra defined by congruence $\sim_p$ is algebra defined by the type p.

Lemma

Let U be a universally axiomatized theory in $\mathcal{L}$. Then for any finitely generated algebra $\mathcal{A}$ in the language $\mathcal{L}$ the following conditions are equivalent:

1) $\mathcal{A} \in \mathrm{Mod}(U)$;
2) $\mathcal{A}$ *is defined by some complete atomic type p in U.*

Notion of local submodel and construction of ultrapower of algebraic structures was introduced by A.I. Maltsev.

In our paper we give definition of limit algebra over an algebra $\mathcal{A}$ as limit of a direct system of local submodels of $\mathcal{A}$.

- A. I. Malcev, *Algebraic structures*, Nauka, Moscow, 1970.
- V. A. Gorbunov, *Algebraic theory of quasivarieties*, Nauchnaya Kniga, Novosibirsk, 1999; English transl., Plenum, 1998.
- E. Daniyarova, A. Miasnikov, V. Remeslennikov, *Unification theorems in algebraic geometry*, Algebra and Discrete Mathematics, **1**, 2008, and on arxiv.org.

Let $\mathcal{A}$ be a finitely generated algebra in a finite language $\mathcal{L}$ and n a positive integer number, $n \geqslant d$, where d is a minimal number of generates for $\mathcal{A}$.

$$\mathrm{AT}_n(\mathcal{A}) = \{\text{atomic type } p \text{ of rank } n \mid p \text{ defines algebra } \mathcal{A}\}.$$

We introduce the structure of ultra-metric space $\langle \mathrm{AT}_n(\mathcal{A}), d \rangle$ with the following metric d:

- $d(p, p) = 0$,
- if $p \neq q$, then $d(p, q) = \frac{1}{2^r}$, where r is the maximal natural number, such that $B_r(p) = B_r(q)$, but $B_{r+1}(p) \neq B_{r+1}(q)$.

Let $B_r(p) = \{\text{all formulas } \Phi \in p \mid c(\Phi) \leqslant r\}$, where $c(\Phi)$ is an Gödel complexity of a formula Φ.

The compactification $\overline{\mathrm{AT}}_n(\mathcal{A})$ for $\mathrm{AT}_n(\mathcal{A})$ is defined standard way by means of Cauchy sequences. $\overline{\mathrm{AT}}_n(\mathcal{A})$ is compact ultra-metric space.

Let $\overline{p}$ be a point in $\overline{\mathrm{AT}}_n(\mathcal{A})$ and $\{g_i\}$ Cauchy sequence, such that $\underrightarrow{\{g_i\}} = \overline{p}$. In standard way one can interprets $\overline{p}$ as atomic type in $\mathrm{AT}_n(-)$ and prove that this type isn't depends on choice of Cauchy sequence $\underrightarrow{\{g_i\}}$. Thus, one can interprets every point from $\overline{\mathrm{AT}}_n(\mathcal{A})$ as atomic type from $\mathrm{AT}_n(-)$.

Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra. Then every algebra, which is defined by atomic type from $\overline{\mathrm{AT}}_n(\mathcal{A})$, belongs to **Ucl**$(\mathcal{A})_\omega$*. And conversely, for each finitely generated an algebra $\mathcal{C}$ in* **Ucl**$(\mathcal{A})$ *there exists an atomic type p from $\overline{\mathrm{AT}}_n(\mathcal{A})$, such that the algebra, defined by p, is isomorphic to $\mathcal{C}$.*

Definition

Algebra defined by an atomic type from $\overline{\mathrm{AT}}_n(\mathcal{A})$ is termed (atomic) limit algebra for $\mathcal{A}$.

Definition

The set $\{\overline{\mathrm{AT}}_n(\mathcal{A}), n \geqslant d\}$ of ultra-metric compact spaces is termed algebraic-geometrical boundary for $\mathcal{A}$.

Let $\mathcal{A}$ be an equationally Noetherian algebra.

Theorem (Finite Length)

Any sequence of proper epimorphisms of limit algebras for $\mathcal{A}$,

$$\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3 \to \ldots,$$

is finite.

Theorem (Finite Width)

Let $\mathcal{B}$ be a finitely generated algebra. Then there exists a finite set of limit algebras $\mathcal{C}_1, \ldots, \mathcal{C}_t$ and epimorphisms

$$\mathcal{B} \to \mathcal{C}_1, \ldots, \mathcal{B} \to \mathcal{C}_t,$$

such that every morphism from $\mathcal{B}$ to the algebra $\mathcal{A}$ factorizes through one of these epimorphisms.

Future

It seems interesting to investigate in details (atomic) limit algebras for the following algebras:

- free Lie algebra over a field;
- free associative algebra;
- free anti-commutative algebra;
- free semigroup;
- finitely generated abelian semigroup;
- free nilpotent group (algebra);
- free solvable group (algebra).