Strong Tits alternative for Coxeter groups

Gennady A. Noskov
SIM, Omsk

"Nine is not a prime," I heard myself say,
certain that I had never said such a thing before.
B- I. Topaz, *Conversations with A. S. Golubitski*

Definition. A group G is large if it virtually has a nonabelian free quotient ($\iff$ there is a subgroup H of finite index in G such that H maps onto a free group of rank 2). (M. Gromov, Volume and bounded cohomology, 1982)

Theorem. ([NV], 2002) *Any subgroup* Γ *of a Coxeter group* W *of finite rank either is large or virtually abelian.*

Case $\Gamma = W$ is crucial:

[MV]: Some linear groups virtually having a free quotient. J. Lie Theory, 10:171–180, 2000.

Tits Alternative (1972) : Every finitely generated linear group either is virtually solvable or contains F_2.

Difference: Γ is large $\Rightarrow$ Γ contains F_2. But not vice versa.

Example. $\Gamma = SL_3(\mathbb{Z})$ contains a free nonabelian group generated by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Γ is not large!

Congruence Subgroup Property (Mennicke):

Every normal subgroup in $SL_3(\mathbb{Z})$ either contains a congruence-subgroup $SL_3(\mathbb{Z}, m\mathbb{Z})$ for $m \neq 0$ or is central, hence finite.

If $SL_3(\mathbb{Z}) \twoheadrightarrow F_2$ then it follows from CSP that the kernel K is finite.

But $SL_3(\mathbb{Z})$ contains

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \mathbb{Z}^2 \tag{1}$$

which maps isomorphically into F_2 - contradiction.

[BFH]:

The Tits alternative for $\mathrm{Out}(F_n)$ I: Dynamics of exponentially-growing automorphisms". Annals of Mathematics 151: 517623.(2000)

The Tits alternative for $\mathrm{Out}(F_n)$ II: A Kolchin type theorem Annals of Mathematics, Vol. 161 , No. 1, 1-59 (2005)

The Tits alternative for $\mathrm{Out}(F_n)$ III: Solvable subgroups are virtually abelian.

Theorem 0.1 (Lackenby, 2005) *Let Γ be a finitely presented group. Then Γ is large if and only if there exists a sequence $\Gamma_1, \Gamma_2 \ldots$ of finite index subgroups of Γ, each normal in Γ_1, such that*

(a) Γ_i/Γ_{i+1} *is abelian for all* $i \geq 1$*;*

(b) $\lim_i((log[\Gamma_i : \Gamma_{i+1}])/[\Gamma : \Gamma_i]) = \infty$*;*

(c) $\limsup_i(d(\Gamma_i/\Gamma_{i+1})/[\Gamma : \Gamma_i]) > 0$*.*

(b) requires the order of Γ_i/Γ_{i+1} to grow super-exponentially in index $[\Gamma : \Gamma_i]$.

(c) asserts that the rank of the quotients Γ_i/Γ_{i+1} grows linearly in the index $[\Gamma : \Gamma_i]$.

Theorem 0.2 (MV) , *2000* *Let $k \subset \mathbb{R}$ be a totally real algebraic number field and f a quadratic form of signature $(n, 1)$ over k that becomes positive definite under every non-identical embedding $\sigma : k \to \mathbb{R}, n \geq 1$. Let A be the ring of integers of k. Then any finitely generated subgroup of $O(f, A)$ either is large or virtually abelian.*

Problem: Is any lattice in $PO_{31} \simeq Isom\left(H^3\right)$ large? Has Γ virtually a $\mathbb{Z}$-quotient?

As important as Poincare Conjecture!

M. Bożejko, T. Januszkiewicz, and R. J. Spatzier (1988):

Infinite Coxeter groups do not have Kazhdan's property.

D. Cooper, D. D. Long, and A. W. Reid (1998) and independently Constantin Gonciulea (1997):

Infinite Coxeter groups are virtually indicable.

Tools:

Dual trees of discrete foliations

Moussong complex of a Coxeter group.

1 Coxeter groups

S - a finite set. A Coxeter matrix (m_{st}) is a symmetric $S \times S$ matrix with entries in $\mathbb{N} \cup \{\infty\}$, 1's on the diagonal, and each off-diagonal entry > 1.

A Coxeter group

$$W = W_S = \langle s : s \in S | (ss')^{m_{ss'}} = 1 \text{ for } m_{ss'} \neq \infty \rangle,$$

W is *spherical* if $|W| < \infty$.

2 Linear representation

$V = \bigoplus_{s \in S} \mathbb{R}s$ – the real vector space with a basis of elements $s \in S$.

A symmetric bilinear form B on $\bigoplus_{s \in S} \mathbb{R}s$ (Tits form)

$$B(s, s') = -\cos(\pi/m_{ss'}).$$

Each s acts as a reflection

$$r_s : v \mapsto sv = v - 2B(v, s)s$$

in the hyperplane $H_s = \mathbb{R}s^{\perp}$.

The map $s \mapsto r_s$ extends to an exact linear representation of W.

3 Building up the Moussong complex

Suppose W is finite. W acts by isometries of Euclidean space (V, B).

For $s \in S, w \in W$ the element $w^{-1}sw$ acts as a reflection. The reflecting hyperplanes cut the space V into simplicial cones.

Choose x_0 in a cone that lies distance $1/2$ from the defining hyperplanes of the cone.

Moussong cell M is convex hull of the orbit Wx_0.

Orbit map $W \to Wx_0$ is one one and moreover $Wx_0 = M_W^{(0)}$.

Label the vertices of M_W by the group elements of W.

For infinite W we have a Moussong cell M_J For each spherical $W_J \leq W$. For $w \in W$ consider a copy (w, M_J) with vertices labelled by a left coset wW_J.

The Moussong complex

$$M_W = \bigcup \{(w, M_J) : w \in W, J \text{ is spherical } \}/ \sim .$$

Two faces are glued via isometry preserving the labels.

M_W is a piecewise Euclidean cell complex i.e. a connected cell complex made up by gluing together Euclidean cells via isometries of their faces (n can vary).

The path metric d on X:

$$d(x, y) = \text{ infimum of the lengths of polygonal paths joining } x \text{ to } y.$$

If there are only finitely many isometry types of cells in X, the path metric is a complete, geodesic metric.

Curvature. Let Δ be a geodesic triangle in X. A comparison triangle for Δ is a Euclidean triangle Δ' with the same side lengths as Δ.

M_W is nonpositively curved, (or CAT(0)-space).

G. Moussong (1988, PhD thesis):

For any Coxeter group W of finite rank there is a complete, contractible, locally finite piecewise Euclidean complex M_W of nonpositive curvature on which W acts properly and cocompactly by isometries. There are only finitely many isometry types of cells in M_W, in particular M_W is complete and geodesic relative to the path metric.

Examples.

- W = dihedral group of order $2m$,

  $\mathcal{M}$ = regular $2m$-gon with the natural W-action.

- Permutahedron: S_n acts on a hyperplane $\sum x_n$.

- W = affine Coxeter group of rank n,

  M_W = a tesselation of the $n-1$-dimensional Euclidean space E.

  This tesselation is dual to the tesselation, given by the structure of Coxeter complex on E.

- $W =< s_1, s_2, s_3 >$, where s_i- reflections in the sides of an equilateral triangle C of a Euclidean plane. Then M_W is a tesselation of the plane by the hexagons, dual the tesselation $W \cdot C$.

- If $W = *_n(\mathbb{Z}/2), n \geq 2$ (that is $m_{ss'} = \infty$ for $s \neq s'$), then $\mathcal{M}_W$ is an infinite n-regular tree with edges of length 1.

Walls in M_W = the fix point sets of reflections in W.

Lemma 3.1 *Any geodesic in M_W having at least two distinct points in a wall H, entirely lies in H.*

Each wall divides M_W into two connected convex components. The walls yield a partition of M_W into convex sets. The closures of these sets are called **chambers**.

All chambers of M_W with an appropriate adjacency relation constitute the Cayley graph M_W. The Cayley graph distance between two chambers is equal to the number of walls separating these chambers.

Isometries of M_W. For $g \in W$ $\quad d_g : M_W \to \mathbb{R}_+ =$ displacement function

$$d_g(x) = d(gx, x).$$

$|g| := \inf\{d_g(x) : x \in M_W\}-$ translation length of g.

$\mathrm{Min}(g)$- set of points where d_g attains infimum.

If g is non-periodic there is a geodesic axis A_g in M_W, isometrical to $\mathbb{R}$, on which g acts by translation.

$\forall x \in \mathrm{Min}(g) \qquad A_g = \cup_{n\in\mathbb{Z}}[g^n x, g^{n+1}x].$

If C is a closed convex g-invariant subset then $C \cap \mathrm{Min}(g)$ is nonempty. In particular there is an axis $A_g \subset C$.

4 Dual trees for wall foliations

Lemma 4.1 *Let W be a Coxeter group of finite rank and let W_0 be a normal torsionfree finite index subgroup in W. Then W_0 has* trivial intersection property, *i.e. for any $g \in W_0$ and any wall H either $gH = H$ or $gH \cap H = \emptyset$.*

For $\Gamma \leq W$ and a wall H the walls $gH (g \in \Gamma)$ yield a partition of W into convex sets, which we shall call Γ-chambers.

Graph $\mathcal{T} = \mathcal{T}(\Gamma, H)$,

Vertices = Γ-chambers

Edges = the walls $gH (g \in \Gamma)$

An edge gH connects two vertices if the corresponding regions share the wall gH.

The action is transitive on the set of edges.

5 Residually finite actions

An action of a group Γ on a set X is called **residually finite**, if for any two distinct elements $x, x' \in X$ there is an action of Γ on a finite set F and a Γ-equivariant map $f : X \to F$ such that $f(x) \neq f(x')$.

Proposition 5.1 *Let* Γ *be a subgroup in a non-affine indecomposable Coxeter group* W *of finite rank and let* H *be a wall such that for any* $g \in \Gamma$ *either* $gH = H$ *or* $gH \cap H = \emptyset$*. Then the action of* $\Gamma : E\mathcal{T}(\Gamma, H)$ *is residually finite.*

6 Strong Tits alternative

Assume W is not affine or finite, Γ is torsionfree and that for any wall H the trivial intersection property holds true, i.e. either $gH = H$ or $gH \cap H = \emptyset$ for any $g \in \Gamma$.

Lemma 6.1 *Let $g \in \Gamma$ be a hyperbolic element with an axis A_g and let H be a wall transversal to A_g. Then $\mathcal{T}$ is not a* star*, i. e. is there is no vertex incident to all the edges of $\mathcal{T}$.*

We can already prove STA in the case when there is a wall H such that the tree $\mathcal{T} = \mathcal{T}(\Gamma, H)$ is not a line. Indeed, possibly passing to a subgroup of index 2, we may assume that Γ does not reverse edges

Proposition 6.2 *Let a group Γ act on a tree $\mathcal{T}$ which is not a star or a line. Suppose that Γ does not reverse edges and the action on the edges is transitive and residually finite. Then Γ is large.*

Thus, throughout, assume that

- *The tree $\mathcal{T} = \mathcal{T}(\Gamma, H)$ is a line for any wall H, such that H is transversal to an axis of some element in Γ.*

Start an approximation process. $g_0 \in \Gamma$ - an arbitrary nontrivial and let H_0 - a wall transversal to some axis A_{g_0} of g_0. Assume inductively that the pairwise distinct walls $H_0, H_1, \dots, H_j$ have been constructed so that each $H_i, i \leq j$ does not contain $H_0 \cap H_1 \cap \dots \cap H_{i-1}$.

If the group

$$\Gamma_j = \bigcap_{i=0}^{j} \mathrm{Stab}_\Gamma(H_i)$$

is nontrivial then take $1 \neq g_{j+1} \in \Gamma_j$ and choose the axis $A_{g_{j+1}}$ for this element to be lying entirely in in $H_0 \cap H_1 \cap \dots \cap H_j$. (Such one exists since $H_0 \cap H_1 \cap \dots \cap H_j$ is convex, closed and g_{j+1}-invariant.) Then choose the wall H_{j+1}, transversal to $A_{g_{j+1}}$.

The process must terminate in view of the following

Lemma 6.3 *The codimension of the intersection of all H_i's is greater or equals k.*

Proof. $\forall i, H_i$ does not contain $H_0 \cap H_1, \ldots, \cap H_{i-1}$. The same is true for the intersections of H_i's with any cell C containing a common point with the intersection of all H_i's (since the reflection r_i in H_i is not contained in the subgroup generated by $r_0, r_1, \ldots, r_{i-1}$. Hence, the codimension of the intersection of all H_i's with C is at least k. □

It follows that $\exists k \geq 1$ such that $\cap_{i=0}^{k} \mathrm{Stab}_{\Gamma}(H_i) = 1$.

A sequence of trees $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k$, corresponding to Γ and to the walls $H_0, H_1, \ldots, H_k$.

All the trees are lines. May assume passing to the finite index subgroup in Γ that the action of Γ on each of the trees $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k$ is by translation.

Have obtained a homomorphism $\pi : \Gamma \to \mathbb{Z}^{k+1}$. This homomorphism is injective.

Indeed, if $g \in \Gamma$ acts trivially on each of the trees above then it stabilizes all the walls $H_0, H_1, \ldots, H_k$, hence $g = 1$. STA is proved.

7 Conclusion

We presented a group-theoretic notion of largeness due to M. Gromov. We gave a sketch of the proof of STA for Coxeter groups which asserts that any subgroup in a Coxeter group of finite rank is either large or virtually abelian. It follows that large groups exist and sometimes they are very interesting.

The main tools in the proof are the dual trees of discrete foliations and Moussong complex of a Coxeter group. The intuitive idea of Moussong complexes was given with examples and pictures.

In a forthcoming paper, we plan to extend these ideas to group actions on affine buildings.