Geometrical and universal geometrical equivalencies for partially commutative nilpotent groups

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Let R be a Abelian domain of integrity contains $\mathbb{Z}$ as a subring. We call R binomial ring, if for every $\lambda \in R$ and for every natural number n, the ring R contains the binomial coefficient:

$$C_\lambda^n = \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)}{n!}.$$

Definition

Let G be a nilpotent group of class m and R be a binomial ring. We call group G R-group, if for every $\lambda \in R$ and $x \in G$ there is unique element x^λ in group G and on every elements $x, y, x_1, \ldots, x_n \in G$ and $\lambda, \mu \in R$ the following axioms hold:

1. $x^1 = x,\ x^\lambda x^\mu = x^{\lambda+\mu},\ (x^\lambda)^\mu = x^{\lambda\mu}$.
2. $y^{-1}x^\lambda y = (y^{-1}xy)^\lambda$.
3. $x_1^\lambda \ldots x_n^\lambda = (x_1, \ldots x_n)^\lambda \tau_2^{C_\lambda^2}(X) \ldots \tau_m^{C_\lambda^m}(X)$, where $X = \{x_1, \ldots, x_n\}$, $\tau_i(X)$ – i-th Petrescu's word.

Recall that for every $i \in \mathbb{N}$, i-th Petrescu's word is defined by the following recursive formula:

$$x_1^i \ldots x_n^i = \tau_1^{C_i^1}(X)\tau_2^{C_i^2}(X)\ldots\tau_{i-1}^{C_i^{i-1}}(X)\tau_i^{C_i^i}(X)$$

in a free group F with generators $x_1, \ldots, x_n$, in particular, $\tau_1(X) = x_1 x_2 \ldots x_n,\ \tau_2(X) = \prod_{i<j,\ i,j=1}^{n} [x_i, x_j] \bmod \gamma_3(F)$, where $\gamma_3(F)$ is a third element of lower central series of group F.

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In this talk we use nilpotent groups of class $m = 2$.

The variety $\mathfrak{N}_2$ of nilpotent groups of class 2 is defined by the following indentity:

$$\forall x, y, z \ [[x, y], z] = 1.$$

In $\mathfrak{N}_2$ we rewrite the axiom №3 from definition of R-group as follow:

3$'$. $x_1^\lambda \dots x_n^\lambda = (x_1, \dots x_n)^\lambda \tau_2^{C_\lambda^2}(X)$, *where* $\tau_2(x_1, \dots, x_n) = \prod_{i<j,\ i,j=1}^{n} [x_i, x_j]$.

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We denote by $\mathfrak{N}_{2,R}$ the variety of nilpotent R-group of class 2.

Let $F_{n,R}$ be a free group in variety $\mathfrak{N}_{2,R}$ with base $V = \{a_1, \dots, a_n\}$. Let Γ be a finite simple (undirected, with no loops and multiple edges) graph with the set of vertices $V(\Gamma) = V$ and the set of edges $E(\Gamma)$. We define the partially commutative nilpotent R-group of class 2 in variety $\mathfrak{N}_{2,R}$ as follow:

$$G_\Gamma = \langle V | R_\Gamma \rangle_{\mathfrak{N}_{2,R}}, \text{ where } R_\Gamma = \{[a_i, a_j] = 1 | \forall (a_i, a_j) \in E(\Gamma)\}.$$

In this talk we consider ring R as a rational numbers field $\mathbb{Q}$. And then we denote G_Γ as a partially commutative nilpotent $\mathbb{Q}$-group of class 2.

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Let G be a nilpotent R-group of class 2. A Cartesian power $G^n = G \times \ldots \times G$ (n copies) is referred to as an affine space over G.

Let $X = \{x_1, \ldots, x_n\}$ be a set of letters and denote by $G[X]$ a nilpotent product $G *_{\mathfrak{N}_{2,R}} F_{2,R}(X)$, where $F_{2,R}(X)$ is a free nilpotent R-group of class 2 with base X.

A system S of equations over G is a subset in $G[X]$. An element $u \in S$ can be conceived of as a noncommutative polynomial in variables $x_1, \ldots, x_n$ with coefficients from G. An element $p = (g_1, \ldots, g_n) \in G^n$ is called a root of a polynomial $u = u(x_1, \ldots, x_n)$ if $u(g_1, \ldots, g_n) = 1$ in G.

Let S be a subset in $G[X]$; then p is called a root of S if p is a root for every $u \in S$.

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Let S be a subset in $G[X]$; then p is called a root of S if p is a root for every $u \in S$.

A subset V in the affine space G^n is called an algebraic set over G if V is the set of all solutions to a system of equations $S \subseteq G[X]$.

The radical of S is defined thus:

$$\mathrm{Rad}(V) = \{u \in G[X] | u(p) = 1 \text{ for all } p \in V\}.$$

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Let L be a group language with no constants.

Definition

Groups A and B are said to be geometrically equivalent in L if for any natural number n and for any system S of equations in n variables holds:

$$\mathrm{Rad}_A(S) = \mathrm{Rad}_B(S).$$

In other words, the geometrical equivalence of two groups means that finding a solution for a system of equations in one group is equivalent to finding a solution for the same system in the other group.

Theorem 1

Let G_{Γ_1} and G_{Γ_2} be two non-abelian partially commutative nilpotent $\mathbb{Q}$-groups of class 2. Then G_{Γ_1} and G_{Γ_2} are geometrically equivalent.

In proof we use two results:

Lemma (coefficient-free case)

Two algebras B and C are geometrically equivalent if and only if

$$\mathrm{pvar}(B)_\omega = \mathrm{pvar}(C)_\omega .$$

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Theorem (coefficient-free case) [A. Miasnikov, V. Remeslennikov]

Let a group G be equationally Noetherian. Then, for any finitely generated group H, the conditions below are equivalent:

- H is the coordinate group of some non-empty algebraic set over G;
- H is separated by G ($H \in \mathrm{Res}(G)$);
- $H \in \mathrm{pvar}(G)$;
- $H \in \mathrm{qvar}(G)$.

Definition

Two groups A and B are said to be universally geometrically equivalent in a language L if, for any natural n and for any system S of equations in n variables,

$$\mathrm{Rad}_A(S) = \mathrm{Rad}_B(S),$$

and the radical $\mathrm{Rad}_A(S)$ is irreducible over A if and only if $\mathrm{Rad}_B(S)$ is irreducible over B.

Theorem 2

Two partially commutative nilpotent $\mathbb{Q}$-groups of class 2 G_{Γ_1} and G_{Γ_2} are universally geometrically equivalent if and only if $\Phi(G_{\Gamma_1}) = \Phi(G_{\Gamma_2})$.

Here, $\Phi(G_\Gamma)$ is a set of existential formulas $\phi(T)$ of special form, hold on G_Γ.

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Here, $\Phi(G_\Gamma)$ is a set of existential formulas $\phi(T)$ of special form, hold on G_Γ.

Let T be a finite simple (undirected, with no loops and multiple edges) graph with vertex set $X = \{x_1, \ldots, x_n\}$. We define existential formula $\phi(T)$ corresponding to graph T in the following way:

$$\phi(T) = \exists x_1, \ldots, x_n (\bigwedge_{i,j}[x_i, x_j] = 1 \wedge \bigwedge_{k,l}[x_k, x_l] \neq 1 \wedge \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i=\overline{1,n}} x_i \neq 1)$$

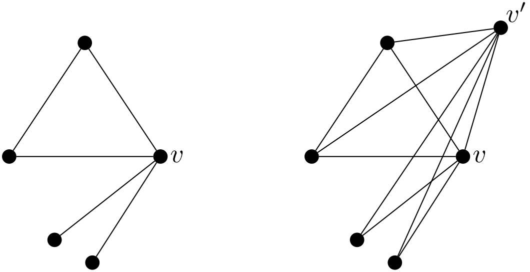
Denote by $\Phi(G)$ the set of all formulas $\phi(T)$ which hold on group G.

Theorem [A. Mishchenko, A. Treyer]

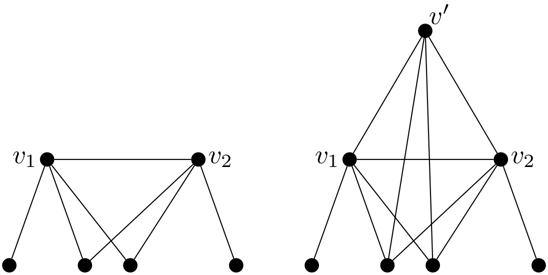
Let G_Γ be a partially commutative nilpotent $\mathbb{Q}$-group of class 2. The existential formula $\phi(T)$ holds on G_Γ for arbitrary graph T if and only if there exists graph Γ_0 that the next conditions hold:

- Γ_0 is product of applying sequence of elementary inflations and deflations of first, second or third type to graph Γ;
- T is full subgraph of Γ_0.

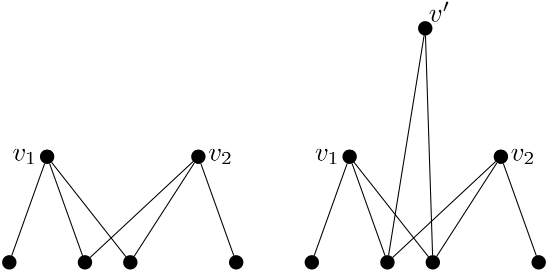
First type inflation



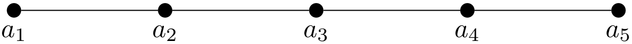
Second type inflation



Third type inflation



In particular case when graph Γ is a path of length $n-1$ ($n-1$ – edges, n – vertices) we denote group G_Γ as G_n.



Path with length 4.

Theorem [A. Mishchenko, A. Treyer]

The existential formula $\phi(T)$ holds on G_n for arbitrary graph T if and only if there exists graph T' that the next conditions hold:

- T' is product of full deflation first or second type of graph T;
- $T' \in \overline{Path_k}$, where $k < n$.

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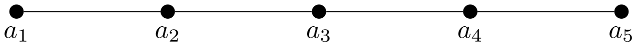

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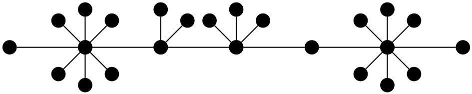
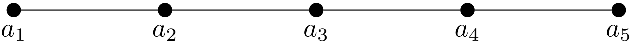
$\overline{Path}_k$ is a class of graphs obtained from path length k by adding pendant vertices to internal vertices.



Examples

Let T be a graph path 4:



And Γ be a graph path 3:

Formula $\phi(T)$ does not hold on group G_Γ ($G_\Gamma = G_4$)
$\Phi(G_4) \neq \Phi(G_5)$
Groups G_4 and G_5 are not universally geometrically equivalent (Theorem 2).
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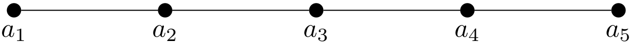

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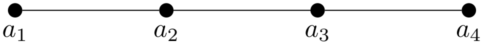

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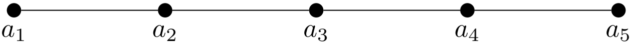

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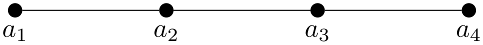

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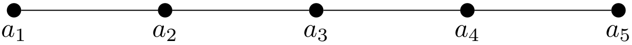

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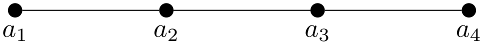

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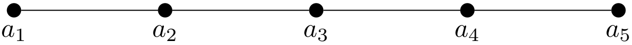

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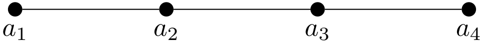

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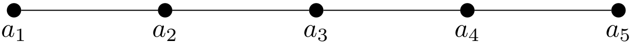

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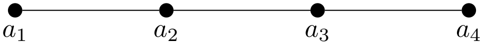

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