

# Geometrical and universal geometrical equivalencies for partially commutative nilpotent groups

Alexei Mishchenko

Sobolev Institute of Mathematics of the SB RAS, Omsk Branch, Russia

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Let  $R$  be a Abelian domain of integrity contains  $\mathbb{Z}$  as a subring. We call  $R$  **binomial ring**, if for every  $\lambda \in R$  and for every natural number  $n$ , the ring  $R$  contains the binomial coefficient:

$$C_{\lambda}^n = \frac{\lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)}{n!}.$$

## Definition

Let  $G$  be a nilpotent group of class  $m$  and  $R$  be a binomial ring. We call group  $G$   **$R$ -group**, if for every  $\lambda \in R$  and  $x \in G$  there is unique element  $x^\lambda$  in group  $G$  and on every elements  $x, y, x_1, \dots, x_n \in G$  and  $\lambda, \mu \in R$  the following axioms hold:

- 1  $x^1 = x$ ,  $x^\lambda x^\mu = x^{\lambda+\mu}$ ,  $(x^\lambda)^\mu = x^{\lambda\mu}$ .
- 2  $y^{-1} x^\lambda y = (y^{-1} x y)^\lambda$ .
- 3  $x_1^\lambda \dots x_n^\lambda = (x_1 \dots x_n)^\lambda \tau_2^{C^\lambda}(X) \dots \tau_m^{C^\lambda}(X)$ , where  $X = \{x_1, \dots, x_n\}$ ,  $\tau_i(X)$  –  $i$ -th Petrescu's word.

Recall that for every  $i \in \mathbb{N}$ ,  $i$ -th Petrescu's word is defined by the following recursive formula:

$$x_1^i \dots x_n^i = \tau_1^{C_i^1}(X) \tau_2^{C_i^2}(X) \dots \tau_{i-1}^{C_i^{i-1}}(X) \tau_i^{C_i^i}(X)$$

in a free group  $F$  with generators  $x_1, \dots, x_n$ , in particular,

$$\tau_1(X) = x_1 x_2 \dots x_n, \quad \tau_2(X) = \prod_{i < j, i, j=1}^n [x_i, x_j] \text{ mod } \gamma_3(F), \text{ where}$$

$\gamma_3(F)$  is a third element of lower central series of group  $F$ .

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In this talk we use nilpotent groups of class  $m = 2$ .

The variety  $\mathfrak{N}_2$  of nilpotent groups of class 2 is defined by the following identity:

$$\forall x, y, z \quad [[x, y], z] = 1.$$

In  $\mathfrak{N}_2$  we rewrite the axiom  $\mathbb{N}^3$  from definition of  $R$ -group as follow:

$$\mathfrak{3}' \cdot x_1^\lambda \dots x_n^\lambda = (x_1, \dots, x_n)^\lambda \tau_2^{C_2^\lambda}(X), \text{ where } \tau_2(x_1, \dots, x_n) = \prod_{\substack{i < j, \\ i, j = 1}}^n [x_i, x_j].$$

We denote by  $\mathfrak{N}_{2,R}$  the variety of nilpotent  $R$ -group of class 2.

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We denote by  $\mathfrak{N}_{2,R}$  the variety of nilpotent  $R$ -group of class 2.

Let  $F_{n,R}$  be a free group in variety  $\mathfrak{N}_{2,R}$  with base  $V = \{a_1, \dots, a_n\}$ .  
Let  $\Gamma$  be a finite simple (undirected, with no loops and multiple edges) graph with the set of vertices  $V(\Gamma) = V$  and the set of edges  $E(\Gamma)$ .  
We define the **partially commutative nilpotent  $R$ -group of class 2** in variety  $\mathfrak{N}_{2,R}$  as follow:

$$G_\Gamma = \langle V | R_\Gamma \rangle_{\mathfrak{N}_{2,R}}, \text{ where } R_\Gamma = \{[a_i, a_j] = 1 | \forall (a_i, a_j) \in E(\Gamma)\}.$$

In this talk we consider ring  $R$  as a rational numbers field  $\mathbb{Q}$ . And then we denote  $G_\Gamma$  as a partially commutative nilpotent  $\mathbb{Q}$ -group of class 2.

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Let  $G$  be a nilpotent  $R$ -group of class 2. A Cartesian power  $G^n = G \times \dots \times G$  ( $n$  copies) is referred to as an **affine space** over  $G$ .

Let  $X = \{x_1, \dots, x_n\}$  be a set of letters and denote by  $G[X]$  a nilpotent product  $G *_{\mathfrak{N}_{2,R}} F_{2,R}(X)$ , where  $F_{2,R}(X)$  is a free nilpotent  $R$ -group of class 2 with base  $X$ .

A **system  $S$  of equations** over  $G$  is a subset in  $G[X]$ . An element  $u \in S$  can be conceived of as a noncommutative polynomial in variables  $x_1, \dots, x_n$  with coefficients from  $G$ . An element  $p = (g_1, \dots, g_n) \in G^n$  is called a **root of a polynomial**  $u = u(x_1, \dots, x_n)$  if  $u(g_1, \dots, g_n) = 1$  in  $G$ .

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A subset  $V$  in the affine space  $G^n$  is called an **algebraic set over  $G$**  if  $V$  is the set of all solutions to a system of equations  $S \subseteq G[X]$ .

The **radical of  $S$**  is defined thus:

$$\text{Rad}(V) = \{u \in G[X] \mid u(p) = 1 \text{ for all } p \in V\}.$$

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Let  $L$  be a group language with no constants.

### Definition

Groups  $A$  and  $B$  are said to be **geometrically equivalent** in  $L$  if for any natural number  $n$  and for any system  $S$  of equations in  $n$  variables holds:

$$\text{Rad}_A(S) = \text{Rad}_B(S).$$

In other words, the geometrical equivalence of two groups means that finding a solution for a system of equations in one group is equivalent to finding a solution for the same system in the other group.

## Theorem 1

Let  $G_{\Gamma_1}$  and  $G_{\Gamma_2}$  be two non-abelian partially commutative nilpotent  $\mathbb{Q}$ -groups of class 2. Then  $G_{\Gamma_1}$  and  $G_{\Gamma_2}$  are geometrically equivalent.

In proof we use two results:

Lemma (coefficient-free case)

Two algebras  $B$  and  $C$  are geometrically equivalent if and only if

$$\text{pvar}(B)_\omega = \text{pvar}(C)_\omega.$$

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### Theorem (coefficient-free case) [A. Miasnikov, V. Remeslennikov]

Let a group  $G$  be equationally Noetherian. Then, for any finitely generated group  $H$ , the conditions below are equivalent:

- $H$  is the coordinate group of some non-empty algebraic set over  $G$ ;
- $H$  is separated by  $G$  ( $H \in \text{Res}(G)$ );
- $H \in \text{pvar}(G)$ ;
- $H \in \text{qvar}(G)$ .

## Definition

Two groups  $A$  and  $B$  are said to be **universally geometrically equivalent** in a language  $L$  if, for any natural  $n$  and for any system  $S$  of equations in  $n$  variables,

$$\text{Rad}_A(S) = \text{Rad}_B(S),$$

and the radical  $\text{Rad}_A(S)$  is irreducible over  $A$  if and only if  $\text{Rad}_B(S)$  is irreducible over  $B$ .

## Theorem 2

Two partially commutative nilpotent  $\mathbb{Q}$ -groups of class 2  $G_{\Gamma_1}$  and  $G_{\Gamma_2}$  are universally geometrically equivalent if and only if  $\Phi(G_{\Gamma_1}) = \Phi(G_{\Gamma_2})$ .

Here,  $\Phi(G_{\Gamma})$  is a set of existential formulas  $\phi(T)$  of special form, hold on  $G_{\Gamma}$ .

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Here,  $\Phi(G_{\Gamma})$  is a set of existential formulas  $\phi(T)$  of special form, hold on  $G_{\Gamma}$ .

Let  $T$  be a finite simple (undirected, with no loops and multiple edges) graph with vertex set  $X = \{x_1, \dots, x_n\}$ . We define **existential formula**  $\phi(T)$  corresponding to graph  $T$  in the following way:

$$\phi(T) = \exists x_1, \dots, x_n \left( \bigwedge_{i,j} [x_i, x_j] = 1 \wedge \bigwedge_{k,l} [x_k, x_l] \neq 1 \wedge \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i=\overline{1,n}} x_i \neq 1 \right)$$

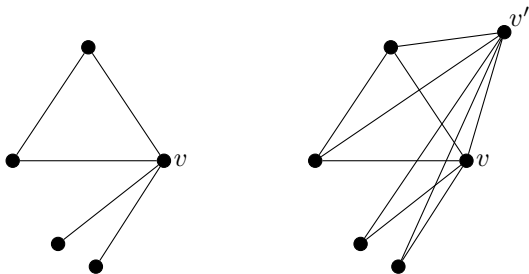
Denote by  $\Phi(G)$  the set of all formulas  $\phi(T)$  which hold on group  $G$ .

### Theorem [A. Mishchenko, A. Treyer]

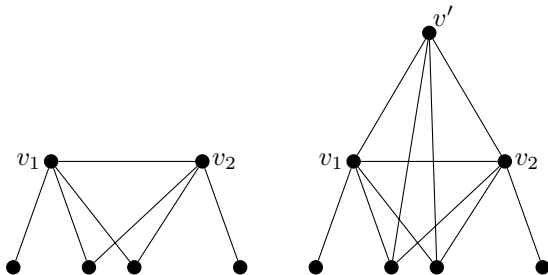
Let  $G_\Gamma$  be a partially commutative nilpotent  $\mathbb{Q}$ -group of class 2. The existential formula  $\phi(T)$  holds on  $G_\Gamma$  for arbitrary graph  $T$  if and only if there exists graph  $\Gamma_0$  that the next conditions hold:

- $\Gamma_0$  is product of applying sequence of elementary inflations and deflations of first, second or third type to graph  $\Gamma$ ;
- $T$  is full subgraph of  $\Gamma_0$ .

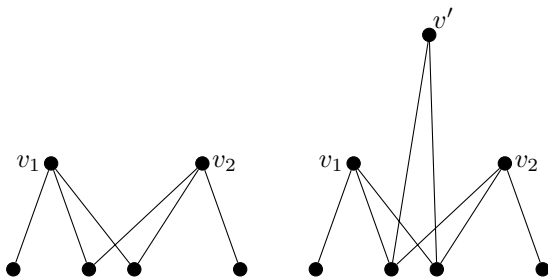
## First type inflation



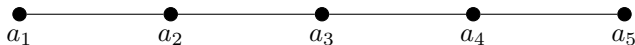
## Second type inflation



## Third type inflation



In particular case when graph  $\Gamma$  is a path of length  $n - 1$  ( $n - 1$  – edges,  $n$  – vertices) we denote group  $G_\Gamma$  as  $G_n$ .



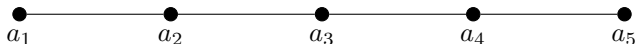
Path with length 4.

Theorem [A. Mishchenko, A. Treyer]

The existential formula  $\phi(T)$  holds on  $G_n$  for arbitrary graph  $T$  if and only if there exists graph  $T'$  that the next conditions hold:

- $T'$  is product of full deflation first or second type of graph  $T$ ;
- $T' \in \overline{Path}_k$ , where  $k < n$ .

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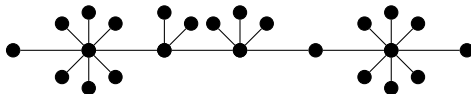
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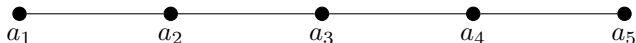
- $T'$  is product of full deflation first or second type of graph  $T$ ;
- $T' \in \overline{Path}_k$ , where  $k < n$ .

$\overline{Path}_k$  is a class of graphs obtained from path length  $k$  by adding pendant vertices to internal vertices.



## Examples

Let  $T$  be a graph path 4:



And  $\Gamma$  be a graph path 3:



Formula  $\phi(T)$  does not hold on group  $G_\Gamma$  ( $G_\Gamma = G_4$ )

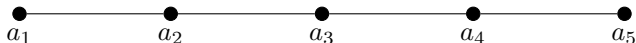
$\Phi(G_4) \neq \Phi(G_5)$

Groups  $G_4$  and  $G_5$  are not universally geometrically equivalent (Theorem 2).

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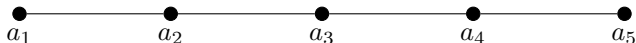
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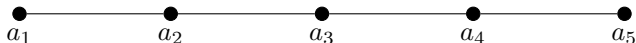
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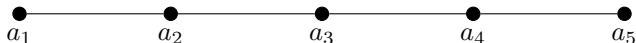
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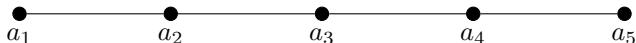
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