

# Automorphisms and identities

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Suppose that  $G$  is a group and  $H$  is its subgroup of finite index. Textbooks in group theory contain some simple facts allowing us to find a finite-index subgroup in  $G$  which is similar to, but better than  $H$ . In particular,

- $H$  contains a normal in  $G$  subgroup of finite index (dividing  $|G : H|!$ );
- if  $G$  is finitely generated, then  $H$  contains a fully invariant (and even verbal) in  $G$  subgroup of finite index;
- if  $H$  is abelian, then  $G$  contains a characteristic abelian subgroup of finite index.

Recently, the last statement was substantially generalised.

**Theorem 1 (Khukhro, Makarenko, 2007).** If a group  $G$  contains a normal finite-index subgroup  $N$  satisfying an outer commutator identity  $w(x_1, \dots, x_t) = 1$ , then  $G$  contains a characteristic finite-index subgroup  $H$  satisfying the same identity and such that

$$\log_2 |G : H| \leq f^{t-1}(\log_2 |G : N|), \quad (\text{Klyachko, Melnikova, 2009})$$

where  $f^k(x)$  is the  $k$ -th iteration of the function  $f(x) = x(x + 1)$ .

**Definition.** Let  $F(x_1, x_2, \dots)$  be a free group of countable rank. An outer commutator of weight 1 is a generator  $x_i$ . An outer commutator of weight  $t > 1$  is a word of the form  $w(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)]$ , where  $u$  and  $v$  are outer commutators of weights  $r$  and  $t - r$ , respectively. Informally, an outer commutator of weight  $t$  is an expression  $[x_1, x_2, \dots, x_t]$  with some arrangement of brackets. An outer commutator identity of weight  $t$  is an identity  $w(x_1, \dots, x_t) = 1$  whose left-hand side is an outer commutator of weight  $t$ .

### Examples.

- solvability:  $[[x_1, x_2], [x_3, x_4]] = 1$  (metabelianity)
- nilpotency:  $[[x_1, x_2], x_3] = 1$  (class 2)
- centre-by-metabelianity:  $[[[x_1, x_2], [x_3, x_4]], x_5] = 1$

**Remark 1.** The condition that the subgroup  $N$  be normal is not essential. It is well known that any finite-index subgroup  $N$  contains a normal finite-index subgroup  $\tilde{N}$  such that  $|G : \tilde{N}|$  does not exceed (and even divides)  $|G : N|!$ . Therefore, Theorem 1 remains valid for non-normal subgroups  $N$ , but with worse estimate

$$\log_2 |G : H| \leq f^{t-1}(\log_2 |G : N|!).$$

**Remark 2.** Group  $G$  contains not only characteristic but even invariant under all surjective endomorphisms subgroup  $H$ .

**Theorem 2 (Khukhro, Klyachko, Makarenko, Melnikova).** Let  $G$  be an algebra (possibly, non-associative) over a field. If  $G$  contains a finite-codimensional subspace  $N$  satisfying a multilinear identity  $w(x_1, \dots, x_t) = 0$ , then  $G$  contains a subspace  $H$  satisfying the same identity, invariant under all surjective endomorphisms, and such that  $\text{codim } H \leq f^{t-1}(\text{codim } N)$ . This subspace  $H$  is left, right, or two-sided ideal if the subspace  $N$  is left, right, or two-sided ideal, respectively.

**Theorem 3 (Khukhro, Klyachko, Makarenko, Melnikova).** If a finite  $p$ -group  $G$  contains a normal subgroup  $N$  satisfying an outer commutator identity  $w(x_1, \dots, x_t) = 1$ , then  $G$  contains a characteristic subgroup  $H$  satisfying the same identity and such that

$$\text{rank } G/H \leq f^{t-1}(\text{rank } G/N).$$

**Definition.**  $\text{rank } G$  is the minimal positive integer  $n$  such that every finitely generated subgroup of  $G$  can be generated by  $n$  elements.



**Theorem 1' (Khukhro, Klyachko, Makarenko, Melnikova).** Let  $w(x_1, \dots, x_t)$  be an outer commutator. Then, in any group, there are only finitely many finite-index subgroups which are maximal (by inclusion) among normal subgroups satisfying the identity  $w(x_1, \dots, x_t) = 1$ . Moreover, the number of such subgroups of index  $\leq n$  does not exceed  $2^{F^{t-1}(n)}$ , where  $F^k(x)$  is the  $k$ -th iteration of the function  $F(x) = xn^{2^x}$ .

**Remark.** This theorem consists of two independent assertions. On the one hand, the number of subgroups of index  $\leq n$  which are maximal among normal subgroups with the identity  $w(x_1, \dots, x_t) = 1$  is bounded by an explicit function of  $n$  and  $t$ . This function grows very fast, but on the other hand, the total number of finite-index subgroups which are maximal among normal subgroups with given identity is finite.

The following theorem shows that a similar statement is valid for subspaces (or ideals) in algebras.

**Theorem 2' (Khukhro, Klyachko, Makarenko, Melnikova).** Let  $w(x_1, \dots, x_t)$  be a multilinear element of the free (non-associative) algebra over a field  $F$ . Then, in any algebra over  $F$ , the intersection of all finite-codimensional ideals which are maximal (by inclusion) among all ideals satisfying the identity  $w(x_1, \dots, x_t) = 0$  has finite codimension. Moreover, the intersection of such ideals of codimension  $\leq n$  has codimension not larger than some number depending only on  $n$  and  $t$ . Here, the word "ideal" means left, right, two-sided ideal, or simply subspace (zero-side ideal).



**Theorem (folklore).** Any extension of a solvable of derived length  $s$  group by a solvable of derived length  $t$  group is solvable of derived length  $\leq t + s$ . This estimate is best possible.

**Theorem 5 (Khukhro, Klyachko, Makarenko, Melnikova).** Any extension of a virtually solvable of derived length  $s$  group by a virtually solvable of derived length  $t$  group is virtually solvable of derived length  $\leq t + s + 1$ . This estimate is best possible.

The following theorem shows that Theorem 1 can not be extended to arbitrary identities.

**Theorem 6 (Khukhro, Klyachko, Makarenko, Melnikova).** For any sufficiently large prime  $p$ , there exists a group  $G$  of exponent  $p^2$  with a finite-index subgroup of exponent  $p$ , but without characteristic finite-index subgroups of exponent  $p$ .

Thank you for your  
attention!