Automorphisms and identities

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Suppose that G is a group and H is its subgroup of finite index. Textbooks in group theory contain some simple facts allowing us to find a finite-index subgroup in G which is similar to, but better than H. In particular,

– H contains a normal in G subgroup of finite index (dividing $|G:N|!$);

– if G is finitely generated, then H contains a fully invariant (and even verbal) in G subgroup of finite index;

– if H is abelian, then G contains a characteristic abelian subgroup of finite index.

Recently, the last statement was substantially generalised.

Theorem 1 (Khukhro, Makarenko, 2007). If a group G contains a normal finite-index subgroup N satisfying an outer commutator identity $w(x_1, \ldots, x_t) = 1$, then G contains a characteristic finite-index subgroup H satisfying the same identity and such that

$$\log_2 |G : H| \leq f^{t-1}(\log_2 |G : N|), \quad (Klyachko, Melnikova, 2009)$$

where $f^k(x)$ is the k-th iteration of the function $f(x) = x(x+1)$.

Definition. Let $F(x_1, x_2, \dots)$ be a free group of countable rank. *An outer commutator of weight 1* is a generator x_i. *An outer commutator of weight* $t > 1$ is a word of the form $w(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)]$, where u and v are outer commutators of weights r and $t - r$, respectively. Informally, an outer commutator of weight t is an expression $[x_1, x_2, \dots, x_t]$ with some arrangement of brackets.*An outer commutator identity of weight* t is an identity $w(x_1, \dots, x_t) = 1$ whose left-hand side is an outer commutator of weight t.

Examples.

– solvability: $[[x_1, x_2], [x_3, x_4]] = 1$ (metabelianity)
– nilpotency: $[[x_1, x_2], x_3] = 1$ (class 2)
– centre-by-metabelianity: $[[[x_1, x_2], [x_3, x_4]], x_5] = 1$

Remark 1. The condition that the subgroup N be normal is not essential. It is well known that any finite-index subgroup N contains a normal finite-index subgroup $\widetilde{N}$ such that $|G:\widetilde{N}|$ does not exceed (and even divides) $|G:N|!$. Therefore, Theorem 1 remains valid for non-normal subgroups N, but with worse estimate

$$\log_2 |G:H| \leq f^{t-1}(\log_2 |G:N|!).$$

Remark 2. Group G contains not only characteristic but even invariant under all surjective endomorphisms subgroup H.

Theorem 2 (Khukhro, Klyachko, Makarenko, Melnikova). Let G be an algebra (possibly, non-associative) over a field. If G contains a finite-codimensional subspace N satisfying a multilinear identity $w(x_1, \ldots, x_t) = 0$, then G contains a subspace H satisfying the same identity, invariant under all surjective endomorphisms, and such that $\mathrm{codim\,H} \leqslant \mathrm{f^{t-1}(codim\,N)}$. This subspace H is left, right, or two-sided ideal if the subspace N is left, right, or two-sided ideal, respectively.

Theorem 3 (Khukhro, Klyachko, Makarenko, Melnikova). If a finite p-group G contains a normal subgroup N satisfying an outer commutator identity $w(x_1, \ldots, x_t) = 1$, then G contains a characteristic subgroup H satisfying the same identity and such that

$$\mathrm{rank\ G/H} \leqslant \mathrm{f^{t-1}(rank\ G/N)}.$$

Definition. $\mathrm{rank\,G}$ is the minimal positive integer n such that every finitely generated subgroup of G can be generated by n elements.

Theorem 1′ (Khukhro, Klyachko, Makarenko, Melnikova). Let $w(x_1, \ldots, x_t)$ be an outer commutator. Then, in any group, there are only finitely many finite-index subgroups which are maximal (by inclusion) among normal subgroups satisfying the identity $w(x_1, \ldots, x_t) = 1$. Moreover, the number of such subgroups of index $\leq n$ does not exceed $2^{F^{t-1}(n)}$, where $F^k(x)$ is the k-th iteration of the function $F(x) = xn^{2^x}$.

Remark. This theorem consists of two independent assertions. On the one hand, the number of subgroups of index $\leq n$ which are maximal among normal subgroups with the identity $w(x_1, \ldots, x_t) = 1$ is bounded by an explicit function of n and t. This function grows very fast, but on the other hand, the total number of finite-index subgroups which are maximal among normal subgroups with given identity is finite.

The following theorem shows that a similar statement is valid for subspaces (or ideals) in algebras.

Theorem 2′ (Khukhro, Klyachko, Makarenko, Melnikova). Let $w(x_1, \ldots, x_t)$ be a multilinear element of the free (non-associative) algebra over a field F. Then, in any algebra over F, the intersection of all finite-codimensional ideals which are maximal (by inclusion) among all ideals satisfying the identity $w(x_1, \ldots, x_t) = 0$ has finite codimension. Moreover, the intersection of such ideals of codimension $\leq n$ has codimension not larger than some number depending only on n and t. Here, the word "ideal"means left, right, two-sided ideal, or simply subspace (zero-side ideal).

Theorem (folklore). Any extension of a solvable of derived length s group by a solvable of derived length t group is solvable of derived length $\leq t+s$. This estimate is best possible.

Theorem 5 (Khukhro, Klyachko, Makarenko, Melnikova). Any extension of a virtually solvable of derived length s group by a virtually solvable of derived length t group is virtually solvable of derived length $\leq t+s+1$. This estimate is best possible.

The following theorem shows that Theorem 1 can not be extended to arbitrary identities.

Theorem 6 (Khukhro, Klyachko, Makarenko, Melnikova). For any sufficiently large prime p, there exists a group G of exponent p^2 with a finite-index subgroup of exponent p, but without characteristic finite-index subgroups of exponent p.

Thank you for your attention!