Foundations of algebraic geometry over profinite groups

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Profinite group is the projective limit of spectrum of finite groups.
Profinite topology is naturally defined over profinite groups.
Profinite groups are characterized as compact totally disconnected in the class of topological groups.

When we say about subgroups of profinite groups, we consider only closed subgroup.

Let G be a fixed profinite group.
A profinite group H is termed a G-group if it contains subgroup, which is isomorphic to G. G-subgroup, G-homomorphism and other terms are defined for G-groups.

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Let G be a profinite group and let $< x_1, \ldots, x_n >$ be a free profinite group generated by $\{x_1, \ldots, x_n\}$.
Let $F = G* < x_1, \ldots, x_n >$, where $*$ is a free profinite product.

Definition

Expression $v(x_1, \ldots, x_n) = 1$ is termed equation over G, where $v(x_1, \ldots, x_n) \in F$.

Notice, that in general the left part of equation is not a word over profinite group G on $x_1, \ldots, x_n$.

Definition

The elememt $(g_1, \ldots, g_n) \in G^n$ is the solution of equation $v(x_1, \ldots, x_n) = 1$ if $v(g_1, \ldots, g_n) = 1$, where $v(g_1, \ldots, g_n)$ is the image of $v(x_1, \ldots, x_n)$ under G-epimorphism $F \to G$ defined by mapping $x_i \to g_i$.

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The subset $S \subseteq G^n$ is called the algebraic set if S is the set of solutions of some system of equations on variables $x_1, \ldots, x_n$.

Denote that the set of solutions of equation $v(x_1, \ldots, x_n) = 1$ is closed in profinite topology, since it is the pre-image of 1 under the mapping $G^n \to G$ defined by $(x_1, \ldots, x_n) \mapsto v(x_1, \ldots, x_n)$.

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We define Zariski topology on G^n, where algebraic sets form a prebasis of closed sets.

Let $I(S) \subseteq F$ be an annulator of an algebraic set S.

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Factor group $\Gamma(S) = F/I(S)$ is termed the coordinate group of the set S.

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Any factor group F/H is termed a group of equations if:

1. $H \cap G = 1$;
2. the mapping $x_i \to g_i$ can be continued to G-epimorphism $F/H \to G$.

There exists the maximal subgroup H satisfied these conditions and $F/H = \Gamma(G^n)$.

Theorem 1

Let a profinite group G be the projective limit of spectrum $\mathbb{G} = \{G_i, \varphi^i_j, I\}$ of finite groups G_i, where $\varphi^i_j : G_i \to G_j$ are epimorphisms. Then the groups $\Gamma(G_i^n) = \Gamma_i$ are finite and the canonical epimorphisms $\Gamma_i \to \Gamma_j$ exist $(i \geq j)$ and $\Gamma(G^n) = \varprojlim \Gamma(G_i^n)$.

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A topologic space is termed Noetherian if every properly descending chain of its closed subsets is finite.

Definition

A group G is termed equationally Noetherian if for each natural number n and every system of equations $\{v_i(x_1, \ldots, x_n) = 1 \mid i \in I\}$ there exists a finite subsystem of equations $\{v_i(x_1, \ldots, x_n) = 1 \mid i \in J \subseteq I\}$ with the same set of solutions.

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Definition

A closed set $S \subseteq G^n$ is termed irreducible if $S = S_1 \cup S_2$, where S_1 and S_2 are closed sets, implies that either $S = S_1$ or $S = S_2$.

If G is equationally Noetherian then the affine space G^n is Noetherian and then any closed set $S \subseteq G^n$ can be expressed as a finite union of irreducible algebraic sets $S = S_1 \cup \ldots \cup S_k$, and if expression is uncancelled it is unique.

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Let a profinite group G be a projective limit $\varprojlim G_i$ of finite groups G_i. We denote by $\pi(G) = \cup\pi(G_i)$ the set of prime divisors of orders of groups G_i.

Theorem 2

If the set $\pi(G)$ is infinite then the profinite group G is not equationally Noetherian of single variable.

Theorem 3

If G is a Abelian profinite group and the set $\pi(G)$ is finite then G is equationally Noetherian.

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If G is a Abelian profinite group and the set $\pi(G)$ is finite then G is equationally Noetherian.

Let $\mathbb{Z}_p[[t_\alpha \mid \alpha \in \mathfrak{A}]]$ be a ring of formal serieses of t_α over the ring of p-adic numbers. It is profinite ring and can be expressed as projective limit of factor rings by powers of maximal ideal $M = (p, t_\alpha \mid \alpha \in \mathfrak{A})$. Let I be a closed prime ideal of this ring, R is factor ring by I. The image of M has the same symbol in the ring R. Let $L_m(R)$ be a group of matrices from $GL_m(R)$ comparable with identity matrix by module M. It is pro-p-group.

Definition

Any pro-p-group enclosed in $L_m(R)$ is termed standard linear pro-p-group.

Theorem 4

The group $G = L_m(R)$ and any its subgroup are equationally Noetherian.

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Let A and B be free abelian pro-p-groups with basis $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$.
Let C be a product of A and B in the variety of nilpotent groups of class 2.
H is subgroup C generated by left part of

$$[b_0, a_0] = 1, [b_1, a_0] = [b_1, a_1] = 1, \ldots, \\ [b_n, a_0] = [b_n, a_1] = \ldots = [b_n, a_n] = 1, \ldots, \quad (1)$$

Let's view factor group $D = C/H$ and system of equations

$$[x, a_0] = 1, [x, a_1] = 1, \ldots$$

A finite subsystem of equations with the same set of solutions doesn't exist.
Nilpotent of class 2 pro-p-group D is not equationally Noetherian.

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Let $R = \mathbb{Z}_p[[t, y_1, y_2]]$ be a ring of formal serieses.
$M = (p, t, y_1, y_2)$ is maximal ideal of ring R.
The set of 3×3 -matrices over R comparable with identity matrix by module M is a pro-p-group.
Let $G = \langle c_1, c_2, d_1, d_2 \rangle$, where

$$c_1 = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix},$$

$$d_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, d_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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$$a_0 = c_1, a_1 = [a_0^{-1}, d_1^{-1}], a_2 = [a_1^{-1}, d_1^{-1}], a_3 = [a_2^{-1}, d_1^{-1}], \ldots$$

$$b_0 = c_2, b_1 = [d_2, b_0], b_2 = [d_2, b_1], b_3 = [d_2, b_2], \ldots$$

You can check that

$$a_n = \begin{pmatrix} 1 & 0 & 0 \\ ty_1^n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & ty_2^m & 1 \end{pmatrix}$$

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Notice that the set $\{[a_n, b_m] \mid n, m \in \mathbb{N}\}$ is linear independent over $\mathbb{Z}_p$.

Then group C from previous example is subgroup of group G. Let D and H be groups from previous example too. Since $H \subseteq Z(G)$, then $G/H \geqslant C/H = D$.

Centre-by-metabelian pro-p-group G/H with 4 generators is not equationally Noetherian.

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Thank you for your attention!