Quadratic equation on free monoids

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A brief historical overview

Theorem (Remeslennikov)

No equations $\implies$ *no mathematics*

Equations in groups and monoids

Fix an alphabet of generators A and an alphabet X of variables.

An equation in a free group $F(A)$ is a formal writing

$$W = 1 \quad \text{where } W \in F(A \cup X)$$

An *equation in free monoid* $M(A)$ is a formal writing

$$L = R \quad \text{where } L, R \in M(A \cup X).$$

Systems of equations $\{W_i = 1\}$ and $\{L_i = R_i\}$ are defined in a similar way.

Quadratic equations

A word W is *quadratic* if each variable occurs in W exactly two times. Examples:

$$ax^{-1}bx, \quad x^{-1}y^{-1}xyz^2$$

A set of words $\{W_i\}$ is *quadratic* if each variable occurs in $\{W_i\}$ two times (i.e. the total number of occurrences is 2).

A system of equations $\{W_i = 1\}$ in a group and a system of equations $\{L_i = R_i\}$ in a monoid are *quadratic* if $\{W_i\}$ (resp. $\{L_i, R_i\}$) are quadratic sets of words.

Standard form of quadratic equations in groups

Proposition (A known fact)

Every quadratic equation $W = 1$ in a group can be reduced to one of the following forms:

$$[x_1, y_1][x_2, y_2] \dots [x_g, y_g] = 1 \quad (g \geq 0),$$
$$x_1^2 x_2^2 \dots x_g^2 = 1 \quad (g > 0),$$
$$[x_1, y_1][x_2, y_2] \dots [x_g, y_g] = c_0 z_1^{-1} c_1 z_1 \dots z_m^{-1} c_m z_m \quad (g \geq 0,\ m \geq 0),$$
$$x_1^2 x_2^2 \dots x_g^2 = c_0 z_1^{-1} c_1 z_1 \dots z_m^{-1} c_m z_m \quad (g > 0,\ m \geq 0).$$

Solutions of quadratic equations in free groups

Theorem

Let $W = 1$ be a quadratic equation in a free group F. Then there is a finite number of resolutions describing all solutions of $W = 1$:

$$\begin{array}{ccccccccccc} \overset{U_0}{\circlearrowright} & & \overset{U_1}{\circlearrowright} & & & & \overset{U_{r-1}}{\circlearrowright} & & & & \\ G_0 & \xrightarrow{\phi_1} & G_1 & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_{r-1}} & G_{r-1} & \xrightarrow{\phi_r} & G_r & \longrightarrow & F \end{array}$$

where

1. $G_0 = G_{W=1} = \langle F * F_X \mid W = 1 \rangle$;
2. $G_i = G_{\{W_i=1\}}$ *for certain quadratic systems* $\{W_i = 1\}$;
3. *The last system is trivial; that is,* $G_r = F * F_Y$ *for some* Y;
4. $U_i = \mathrm{Aut}_F(G_i)$;
5. $r \leq 2g(W) + m(W)$.

Solutions of quadratic equations in free monoids

Theorem

Let $\{L_j = R_j\}$ be a quadratic system in a free monoid M (or in a free monoid M with involution). Then there is a finite number of resolutions describing all of its solutions:

$$\overset{U_0}{\circlearrowright} M_0 \xrightarrow{\phi_1} \overset{U_1}{\circlearrowright} M_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{r-1}} \overset{U_{r-1}}{\circlearrowright} M_{r-1} \xrightarrow{\phi_r} M_r \longrightarrow M$$

where

1. $M_0 = M_{\{L_j = R_j\}}$;
2. $M_i = M_{\{L_j^{(i)} = R_j^{(i)}\}}$ *for certain quadratic systems* $\{L_j^{(i)} = R_j^{(i)}\}$;
3. $M_r = M * M_Y$ *for some finite* Y;
4. U_i *is a certain finitely generated group of monoid automorphisms;*
5. $r \leq O(n^3)$ *where* n *is number of items in* $\{L_j = R_j\}$.

Train tracks

Definition

A *train track* is a finite graph T with a partition of the star $S(v)$ of each vertex v into two subsets $S_+(v)$ and $S_-(v)$.

A *surface train track* has, additionally:

- a cyclic ordering of $S(v)$ so that edges from $S_+(v)$ and $S_-(v)$ are consequent.
- for each edge, a choice if it is side-preserving or side-reversing