The averaged Dehn function relative to a given probability measure

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- Dehn (1910–1912) proved that the word problem for the standard presentation of the fundamental group of a closed oriented surface of genus at least two is solvable by what is now called Dehn's algorythm. The direct consequence of this fact is that for this presentation the Dehn function satisfies $Dehn(n) \preceq n$.
- Gromov (1987) introduced the formal notion of an isoperimetric function and a Dehn function in his monograph "Hyperbolic groups" .

Let $G = F_n/R$ be f.p. group. $P(G) =< x_1, \ldots, x_r | r_1, \ldots, r_m >$ is its corresponding presentation.
Clearly, the equality $w =_G 1$ is equivalent to $w =_{F_r} g_1 g_2 \ldots g_s$, where $g_i = \left(r_{i_j}^{\varepsilon_i}\right)^{f_i}$.
The *area* S(w) of a word $w =_G 1$ is least s for which we have this expression.

Definition

The *Dehn function* of G is the function

$$D(n) = \max_{w =_G 1, \, |w| \leq n} S(w).$$

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- Gersten S.M., Holt D.F. and Riley T.R. (2003): Every finitely generated nilpotent group G admits a polynomial isoperimetric inequality of degree $c+1$, where c is the nilpotency class of G.

Define $\Omega_n = \{w =_G 1 | |w| \leq n\}$, the set of words in alphabet X_r whose values in G equal 1 and whose length does not exceed a given number n.
The following definition can be found in M.Gromov's article "Asymptotic invariants of infinite groups" (1993)

Definition

The averaged Dehn function of G is the function $\sigma(n) = \frac{\sum\limits_{w \in \Omega_n} S(w)}{|\Omega_n|}$.

Hypothesis

The averaged Dehn function of free abelian groups is subquadratic; i.e.

$$\lim_{n\to\infty} \frac{\sigma(n)}{n^2} = 0.$$

Question

Is averaged Dehn function subasymptotic (i.e. $\frac{\sigma(n)}{D(n)} \to 0$) for another groups?

- E. G. Kukina and V. A. Roman'kov (2003): Averaged Dehn function for free abelian groups is subquadratic.

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- R. Young. (2005–2008): If a nilpotent group satisfies the isoperimetric inequality $D(n) \preceq n^c$, $c > 2$, then it satisfies the averaged isoperimetric inequality $\sigma(n) \preceq n^{\frac{c}{2}}$. (But his definition of averaged Dehn function is diffrent!)

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- E.G.Kukina (2003) found the example of group G with sublinear averaged Dehn function.

Classic definition:
We write $f \preceq g$, if there are positive constants a, b, c, d such that

$$f(n) \leq ag(bn) + cn + d$$

for every n. And $f \simeq g$ iff $f \preceq g$ and $g \preceq f$.

New definition:
We write $f \preceq g$, if there are positive constants a, b, d such that

$$f(n) \leq ag(bn) + d$$

for every n. And $f \simeq g$ iff $f \preceq g$ and $g \preceq f$.

- Borovik A.V., Myasnikov A.G., Shpilrain V. *Measuring sets in infinite groups*// Contemp.Math. 2002. V.298.P.21–42.

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Let $\Sigma(X)$ is free monoid generated by $X \cup X^{-1}$.
Call $\gamma : \Sigma(X) \to \mathbb{N}$ a *complexity function* whenever γ has finite fibers: *card* $\Gamma_n < \infty$ for each n, where
$\Gamma_n = \{w \in \Sigma(X) | \gamma(w) = n\}$.
Given some probability measure $\{p_n = P\{\xi = n\}\}$ on $\mathbb{N}$, define the probability measure on $\Sigma(X)$ by

$$\forall w \in \Gamma_n \ p(w) = P(\xi = w) = \pi_n = \frac{p_n}{card \Gamma_n}.$$

Call probability good, if $p_n \neq 0 \, \forall n$.

Definition

Relative averaged Dehn function is the function

$$\zeta(n) = \frac{\sum\limits_{w \in \Omega'_n} p(w)S(w)}{p(\Omega'_n)},$$

where $\Omega'_n = \{w =_G 1 | \gamma(w) = n\}$.

One of most usable examples of complexity function is lenght $(\gamma(w) = |w|)$.

Theorem

Relative averaged Dehn function is bounded below by positive constant, except free group in its natural presentation.

Theorem

Let G is f.p. group and $\delta(n) \preceq k^{\beta}$ and $\{p_k\}$ is good probability with property $M\xi^{\beta} \leq \infty$. Then $\exists C > 0$

$$\zeta(n) \leq C.$$