# Equationally Noetherian property and close properties

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# Equationally Noetherian property

Let $\mathcal{L}$ be a functional language. Let $\mathcal{A}$ be an algebraic structure in the language $\mathcal{L}$.

### Definition 1

An algebraic structure $\mathcal{A}$ is called equationally Noetherian if for any positive integer $n$ and any system of equations $S \subseteq \mathrm{At}_{\mathcal{L}}(x_1, x_2, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}_{\mathcal{A}}(S) = \mathrm{V}_{\mathcal{A}}(S_0)$.

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By **N** denote the class of equationally Noetherian algebras.

# Weakly equationally Noetherian property

## Definition 2

An algebraic structure $\mathcal{A}$ is called weakly equationally Noetherian if for any positive integer $n$ and any system of equations $S \subseteq \mathrm{At}_{\mathcal{L}}(x_1, x_2, \ldots, x_n)$ there exists a finite system $S_0 \subseteq \mathrm{At}_{\mathcal{L}}(x_1, x_2, \ldots, x_n)$ such that $\mathrm{V}_{\mathcal{A}}(S) = \mathrm{V}_{\mathcal{A}}(S_0)$.

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By $\mathbf{N}'$ denote the class of weakly equationally Noetherian algebras.

# $q_\omega$-compactness property

## Definition 3

An algebraic structure $\mathcal{A}$ is called $q_\omega$-compact if for any positive integer $n$, any system of equations $S \subseteq \mathrm{At}_{\mathcal{L}}(x_1, x_2, \ldots, x_n)$, and any equation $c \in \mathrm{Rad}_{\mathcal{A}}(S)$ there exists a finite subsystem $S_c \subseteq S$ such that $c \in \mathrm{Rad}_{\mathcal{A}}(S)$.

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# $\mathrm{q}_\omega$-compactness property

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By $\mathbf{Q}$ denote the class of $\mathrm{q}_\omega$-compact algebras.

# The main result

From Definitions 1, 2, and 3, obviously, we have

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We proved that

$$\mathbf{N} \subsetneqq \mathbf{N}' \quad \text{and} \quad \mathbf{N} \subsetneqq \mathbf{Q}.$$

# Example 1 ($\mathbf{N} \neq \mathbf{N}'$)

Let $M_n = \{x : x \leqslant 0 \vee x \geqslant n\}, n \in \mathbb{N}^*,\ M_\infty = \{x : x \leqslant 0\}$,

$$v_n(x) = \begin{cases} -1 + \frac{1}{2}\operatorname{arctg} x, & n = \infty, \\ -n - 1 + \frac{1}{2}\operatorname{arctg} x, & \text{иначе}, \end{cases}$$

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Consider the following algebra:

$$\mathcal{A}_1 = \langle \mathbb{R}; 0, F_\infty, F_1, F_2, \ldots, F_n, \ldots \rangle,$$

$$F_n(x) = \begin{cases} 0, & x \in M_n, \\ v_n(x), & \text{иначе}. \end{cases}$$

We claim that the algebra $\mathcal{A}_1$ is weakly equationally Noetherian but is not equationally Noetherian.

## Example 2 ($\mathbf{N} \neq \mathbf{Q}$)

Let $g_n \colon \mathbb{N} \to \mathbb{N}$ ($n \in \mathbb{N}^*$) such that

$$g_n(x) := \begin{cases} 2n, & x = 2n+1, \\ 2n+1, & x = 2n, \\ x, & \text{otherwise.} \end{cases}$$

Let $I = \{i_1, \ldots, i_n\} \subset \mathbb{N}^*$, $|I| < \infty$,

$$f_I := g_{i_1} \circ g_{i_2} \circ \ldots \circ g_{i_n}, \quad f_\varnothing := \mathrm{id}.$$

We have

$$f_I \circ f_J = f_{I \triangle J}, \quad f_I \circ f_J = f_J \circ f_I,$$

$$f_I \circ f_I = \mathrm{id}, \quad (f_I)^{-1} = f_I,$$

where $\triangle$ is the symmetric difference.

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$$f_I \circ f_I = \mathrm{id}, \quad (f_I)^{-1} = f_I,$$

where $\triangle$ is the symmetric difference.
Let $\mathcal{A}_2 = \left\langle \mathbb{N}; \{g_n\}_{n \in \mathbb{N}^*} \right\rangle$.
We claim that the algebra $\mathcal{A}_2$ is $\mathrm{q}_\omega$-compact but is not equationally Noetherian.