Limits of relatively hyperbolic groups and Lyndon's completions

Olga Kharlampovich, Alexei Miasnikov
(McGill University)

Omsk, 2009

Outline

In this talk I describe finitely generated groups H universally equivalent (with constants from G in the language) to a given torsion-free relatively hyperbolic group G with free abelian parabolics. It turns out that, as in the free group case, the group H embeds into the Lyndon's completion $G^{\mathbb{Z}[t]}$ of the group G, or, equivalently, H embeds into a group obtained from G by finitely many extensions of centralizers. Conversely, every subgroup of $G^{\mathbb{Z}[t]}$ containing G is universally equivalent to G . Since finitely generated groups universally equivalent to G are precisely the finitely generated groups discriminated by G (H is discriminated by G, i.e. for any finite subset $M \subseteq H$ there exists a homomorphism $\phi : H \to G$ injective on M), the result above gives a description of finitely generated groups discriminated by G.

Relative hyperbolicity

A group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ (parabolic subgroups) if G is finitely presented relative to $\{H_\lambda\}_{\lambda\in\Lambda}$

$$G = \langle X \cup (\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda) | \mathcal{R}\rangle,$$

and there is a constant $L > 0$ such that for any word $W \in X \cup \mathcal{H}$ representing the identity in G we have $\mathrm{Area}^{rel}(W) \leq L||W||$. Let $\mathcal{G}$ be a class of f.g. torsion free relatively hyperbolic groups with free abelian parabolics. Groups from $\mathcal{G}$ are CSA (have malnormal maximal abelian subgroups). They also satisfy the Big Powers condition: if $g = g_1 u_1^{n_1} g_2 \dots u_k^{n_k} g_{k+1}$ and $g_{i+1} u_i g_{i+1}^{-1}$ don't commute with u_{i+1}, then $g \neq 1$.

Relative hyperbolicity

A group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ (parabolic subgroups) if G is finitely presented relative to $\{H_\lambda\}_{\lambda\in\Lambda}$

$$G = \langle X \cup (\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda) | \mathcal{R}\rangle,$$

and there is a constant $L > 0$ such that for any word $W \in X \cup \mathcal{H}$ representing the identity in G we have $\mathrm{Area}^{rel}(W) \leq L||W||$. Let $\mathcal{G}$ be a class of f.g. torsion free relatively hyperbolic groups with free abelian parabolics. Groups from $\mathcal{G}$ are CSA (have malnormal maximal abelian subgroups). They also satisfy the Big Powers condition: if $g = g_1 u_1^{n_1} g_2 \dots u_k^{n_k} g_{k+1}$ and $g_{i+1} u_i g_{i+1}^{-1}$ don't commute with u_{i+1}, then $g \neq 1$.

Relative hyperbolicity

A group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ (parabolic subgroups) if G is finitely presented relative to $\{H_\lambda\}_{\lambda\in\Lambda}$

$$G = \langle X \cup (\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda) | \mathcal{R}\rangle,$$

and there is a constant $L > 0$ such that for any word $W \in X \cup \mathcal{H}$ representing the identity in G we have $\mathrm{Area}^{rel}(W) \leq L||W||$. Let $\mathcal{G}$ be a class of f.g. torsion free relatively hyperbolic groups with free abelian parabolics. Groups from $\mathcal{G}$ are CSA (have malnormal maximal abelian subgroups). They also satisfy the Big Powers condition: if $g = g_1 u_1^{n_1} g_2 \dots u_k^{n_k} g_{k+1}$ and $g_{i+1} u_i g_{i+1}^{-1}$ don't commute with u_{i+1}, then $g \neq 1$.

A group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ (parabolic subgroups) if G is finitely presented relative to $\{H_\lambda\}_{\lambda\in\Lambda}$

$$G = \langle X \cup (\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda) | \mathcal{R}\rangle,$$

and there is a constant $L > 0$ such that for any word $W \in X \cup \mathcal{H}$ representing the identity in G we have $\mathrm{Area}^{rel}(W) \leq L||W||$. Let $\mathcal{G}$ be a class of f.g. torsion free relatively hyperbolic groups with free abelian parabolics. Groups from $\mathcal{G}$ are CSA (have malnormal maximal abelian subgroups). They also satisfy the Big Powers condition: if $g = g_1 u_1^{n_1} g_2 \ldots u_k^{n_k} g_{k+1}$ and $g_{i+1} u_i g_{i+1}^{-1}$ don't commute with u_{i+1}, then $g \neq 1$.

Lyndon's exponential group

In 1960 R. Lyndon introduced a notion of a free $\mathbb{Z}[t]$-group, $F^{\mathbb{Z}[t]}$.
Suppose Γ is a CSA group. The group $\Gamma^{\mathbb{Z}[t]}$ can be defined as a union of the chain of groups

$$\Gamma = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_n < \cdots ,$$

where Γ_k is generated by Γ_{k-1} and formal expressions of the type

$$\{w^\alpha \mid w \in \Gamma_{k-1},\ \alpha \in \mathbb{Z}[t]\}.$$

That is, every element of Γ_k can be viewed as a parametric word of the type

$$w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_m^{\alpha_m},$$

where $m \in \mathbb{N},\ w_i \in \Gamma_{k-1}$, and $\alpha_i \in \mathbb{Z}[t]$.

Lyndon's exponential group

In 1960 R. Lyndon introduced a notion of a free $\mathbb{Z}[t]$-group, $F^{\mathbb{Z}[t]}$. Suppose Γ is a CSA group. The group $\Gamma^{\mathbb{Z}[t]}$ can be defined as a union of the chain of groups

$$\Gamma = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_n < \cdots ,$$

where Γ_k is generated by Γ_{k-1} and formal expressions of the type

$$\{w^\alpha \mid w \in \Gamma_{k-1},\ \alpha \in \mathbb{Z}[t]\}.$$

That is, every element of Γ_k can be viewed as a parametric word of the type

$$w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_m^{\alpha_m},$$

where $m \in \mathbb{N}$, $w_i \in \Gamma_{k-1}$, and $\alpha_i \in \mathbb{Z}[t]$.

Lyndon's exponential group

A. G. Miasnikov and V. Remeslennikov (1996) gave an effective construction of $\Gamma^{\mathbb{Z}[t]}$ in terms of extensions of centralizers.
Let G be a group and $C_G(u) = \langle u \rangle$ a cyclic centralizer of $u \in G$.
An extension of $C_G(u)$ by $\mathbb{Z}[t]$ is defined as the HNN-extension

$$H = \langle\ G, s_j \ \ (j \in \mathbb{N}) \ \mid \ [u, s_j] = [s_j, s_k] = 1 \ \ (j, k \in \mathbb{N})\ \rangle.$$

Observe that $C_H(u) \simeq \mathbb{Z}[t]$.

$\Gamma^{\mathbb{Z}[t]}$ is a union of the infinite chain of groups

$$F = G_0 < G_1 < \cdots < G_n < \cdots,$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i.

From the construction of $\Gamma^{\mathbb{Z}[t]}$ and big power condition in Γ it follows that it is discriminated by Γ. Hence, all subgroups of $\Gamma^{\mathbb{Z}[t]}$ are also Γ-discriminated.

For the case $\Gamma = F$

Theorem. (Kharlampovich, Miasnikov, 98) Given a finite presentation of a finitely generated fully residually free group G one can effectively construct an embedding $\phi : G \to F^{\mathbb{Z}[t]}$ (by specifying the images of the generators of G).

$\Gamma^{\mathbb{Z}[t]}$ is a union of the infinite chain of groups

$$F = G_0 < G_1 < \cdots < G_n < \cdots ,$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i.
From the construction of $\Gamma^{\mathbb{Z}[t]}$ and big power condition in Γ it follows that it is discriminated by Γ. Hence, all subgroups of $\Gamma^{\mathbb{Z}[t]}$ are also Γ-discriminated.
For the case $\Gamma = F$

Theorem. (Kharlampovich, Miasnikov, 98) Given a finite presentation of a finitely generated fully residually free group G one can effectively construct an embedding $\phi : G \to F^{\mathbb{Z}[t]}$ (by specifying the images of the generators of G).

Theorem.(KM) Let $\Gamma \in \mathcal{G}$ and H a finitely generated group discriminated by Γ. Then H embeds into a group obtained from Γ by a finite series of centralizer extensions.

Embeddings

The Embedding Theorem [KM]

Let Γ be a torsion-free relatively hyperbolic group with free abelian parabolics. Finitely generated fully residually Γ groups are precisely finitely generated subgroups of $\Gamma^{Z[t]}$.

All the standard corollaries follow.

Embeddings

The Embedding Theorem [KM]

Let Γ be a torsion-free relatively hyperbolic group with free abelian parabolics. Finitely generated fully residually Γ groups are precisely finitely generated subgroups of $\Gamma^{Z[t]}$.

All the standard corollaries follow.

A direct limit of a directed system of finite partial subgroups of G is called a limit group over G if all products of generators eventually appear in these partial subgroups.

G - a group generated by A,
$F(X)$ - free group on $X = \{x_1, x_2, \ldots x_n\}$.

A **system of equations** $S(X, A) = 1$ in variables X and coefficients from G (viewed as a subset of $G * F(X)$).

A **solution** of $S(X, A) = 1$ in G is a tuple $(g_1, \ldots, g_n) \in G^n$ such that $S(g_1, \ldots, g_n) = 1$ in G.

$V_G(S)$, the set of all solutions of $S = 1$ in G, is called an **algebraic set** defined by S.

Algebraic sets

G - a group generated by A,
$F(X)$ - free group on $X = \{x_1, x_2, \dots x_n\}$.

A **system of equations** $S(X, A) = 1$ in variables X and coefficients from G (viewed as a subset of $G * F(X)$).

A **solution** of $S(X, A) = 1$ in G is a tuple $(g_1, \dots, g_n) \in G^n$ such that $S(g_1, \dots, g_n) = 1$ in G.

$V_G(S)$, the set of all solutions of $S = 1$ in G, is called an **algebraic set** defined by S.

Algebraic sets

G - a group generated by A,
$F(X)$ - free group on $X = \{x_1, x_2, \dots x_n\}$.

A **system of equations** $S(X, A) = 1$ in variables X and coefficients from G (viewed as a subset of $G * F(X)$).

A **solution** of $S(X, A) = 1$ in G is a tuple $(g_1, \dots, g_n) \in G^n$ such that $S(g_1, \dots, g_n) = 1$ in G.

$V_G(S)$, the set of all solutions of $S = 1$ in G, is called an **algebraic set** defined by S.

Algebraic sets

G - a group generated by A,
$F(X)$ - free group on $X = \{x_1, x_2, \ldots x_n\}$.

A **system of equations** $S(X, A) = 1$ in variables X and coefficients from G (viewed as a subset of $G * F(X)$).

A **solution** of $S(X, A) = 1$ in G is a tuple $(g_1, \ldots, g_n) \in G^n$ such that $S(g_1, \ldots, g_n) = 1$ in G.

$V_G(S)$, the set of all solutions of $S = 1$ in G, is called an **algebraic set** defined by S.

The maximal subset $R(S) \subseteq G * F(X)$ with

$$V_G(R(S)) = V_G(S)$$

is the **radical** of $S = 1$ in G.

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the **coordinate group** of $S = 1$.

Solutions of $S(X) = 1$ in $G \Longleftrightarrow$ **G-homomorphisms** $G_{R(S)} \to G$.

The maximal subset $R(S) \subseteq G * F(X)$ with

$$V_G(R(S)) = V_G(S)$$

is the **radical** of $S = 1$ in G.

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the **coordinate group** of $S = 1$.

Solutions of $S(X) = 1$ in G $\Longleftrightarrow$ **G-homomorphisms** $G_{R(S)} \to G$.

The maximal subset $R(S) \subseteq G * F(X)$ with

$$V_G(R(S)) = V_G(S)$$

is the **radical** of $S = 1$ in G.

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the **coordinate group** of $S = 1$.

Solutions of $S(X) = 1$ in $G \iff$ G-**homomorphisms** $G_{R(S)} \to G$.

The following conditions are equivalent:

- G is **equationally Noetherian**, i.e., every system $S(X) = 1$ over G is equivalent to some **finite** part of itself.
- the **Zariski topology** (formed by algebraic sets as a sub-basis of closed sets) over G^n is **Noetherian** for every n, i.e., every proper descending chain of closed sets in G^n is finite.
- Every **chain of proper epimorphisms** of coordinate groups over G is **finite**.

Equationally Noetherian groups

The following conditions are equivalent:

- G is **equationally Noetherian**, i.e., every system $S(X) = 1$ over G is equivalent to some **finite** part of itself.
- the **Zariski topology** (formed by algebraic sets as a sub-basis of closed sets) over G^n is **Noetherian** for every n, i.e., every proper descending chain of closed sets in G^n is finite.
- Every **chain of proper epimorphisms** of coordinate groups over G is **finite**.

Equationally Noetherian groups

The following conditions are equivalent:

- G is **equationally Noetherian**, i.e., every system $S(X) = 1$ over G is equivalent to some **finite** part of itself.
- the **Zariski topology** (formed by algebraic sets as a sub-basis of closed sets) over G^n is **Noetherian** for every n, i.e., every proper descending chain of closed sets in G^n is finite.
- Every **chain of proper epimorphisms** of coordinate groups over G is **finite**.

Irreducible components

If the Zariski topology is **Noetherian** then every algebraic set can be uniquely presented as a finite union of its **irreducible components**:

$$V = V_1 \cup \ldots V_k$$

Recall, that a closed subset V is **irreducible** if it is not a union of two proper closed (in the induced topology) subsets.

Irreducible components

If the Zariski topology is **Noetherian** then every algebraic set can be uniquely presented as a finite union of its **irreducible components**:

$$V = V_1 \cup \ldots V_k$$

Recall, that a closed subset V is **irreducible** if it is not a union of two proper closed (in the induced topology) subsets.

Equationally Noetherian groups: results

Theorem [Bryant, Guba].

Linear groups over a commutative, Noetherian, unitary ring (free groups) are equationally Noetherian.

Theorem [Sela]

Hyperbolic groups are equationally Noetherian.

Theorem [Groves]

Relatively hyperbolic groups with abelian parabolics are equationally Noetherian.

Theorem [Romanovskii]

Free solvable groups of finite rank are equationally Noetherian.

Equationally Noetherian groups: results

Theorem [Bryant, Guba].

Linear groups over a commutative, Noetherian, unitary ring (free groups) are equationally Noetherian.

Theorem [Sela]

Hyperbolic groups are equationally Noetherian.

Theorem [Groves]

Relatively hyperbolic groups with abelian parabolics are equationally Noetherian.

Theorem [Romanovskii]

Free solvable groups of finite rank are equationally Noetherian.

Theorem [Bryant, Guba].

Linear groups over a commutative, Noetherian, unitary ring (free groups) are equationally Noetherian.

Theorem [Sela]

Hyperbolic groups are equationally Noetherian.

Theorem [Groves]

Relatively hyperbolic groups with abelian parabolics are equationally Noetherian.

Theorem [Romanovskii]

Free solvable groups of finite rank are equationally Noetherian.

Equationally Noetherian groups: results

Theorem [Bryant, Guba].

Linear groups over a commutative, Noetherian, unitary ring (free groups) are equationally Noetherian.

Theorem [Sela]

Hyperbolic groups are equationally Noetherian.

Theorem [Groves]

Relatively hyperbolic groups with abelian parabolics are equationally Noetherian.

Theorem [Romanovskii]

Free solvable groups of finite rank are equationally Noetherian.

Theorem A *[No coefficients] Let G be an equationally Noetherian group. Then for a finitely generated group H the following conditions are equivalent:*

1. $\mathrm{Th}_\forall(G) \subseteq \mathrm{Th}_\forall(H)$, *i.e.,* $H \in \mathbf{Ucl}(G)$;
2. $\mathrm{Th}_\exists(G) \supseteq \mathrm{Th}_\exists(H)$;
3. *H embeds into an ultrapower of G;*
4. *H is discriminated by G;*
5. *H is a limit group over G;*
6. *H is defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(G)$;
7. *H is the coordinate group of an irreducible algebraic set over G defined by a system of coefficient-free equations.*

Unification theorem

Theorem B *[With coefficients] Let A be a group and G an A-equationally Noetherian A-group. Then for a finitely generated A-group H the following conditions are equivalent:*

1. $\mathrm{Th}_{\forall,A}(G) \subseteq \mathrm{Th}_{\forall,A}(H)$, *i.e.,* $H \in \mathbf{Ucl}_A(G)$;
2. $\mathrm{Th}_{\exists,A}(G) \supseteq \mathrm{Th}_{\exists,A}(H)$;
3. *H A-embeds into an ultrapower of G;*
4. *H is A-discriminated by G;*
5. *H is a limit group over G;*
6. *H is a group defined by a complete atomic type in the theory $\mathrm{Th}_{\forall,A}(G)$ in the language $Ł_A$;*
7. *H is the coordinate group of an irreducible algebraic set over G defined by a system of equations with coefficients in A.*

A triangular quasi-quadratic (TQ) system has the following form

$$
\begin{aligned}
&S_1(X_1, X_2, \ldots, X_n, A) = 1,\\
&\quad S_2(X_2, \ldots, X_n, A) = 1,\\
&\qquad \ldots\\
&\qquad\quad S_n(X_n, A) = 1
\end{aligned}
$$

where S_i is either quadratic in variables X_i, or corresponds to an extension of a centralizer, or to an abelian extension.

Extension Theorem

A TQ system

$$S_1(X_1, X_2, \dots, X_n, A) = 1,$$
$$S_2(X_2, \dots, X_n, A) = 1,$$
$$\dots$$
$$S_n(X_n, A) = 1$$

is **non-degenerate (NTQ)** if for every i the equation $S_i(X_i, \dots, X_n, A) = 1$ has a solution in the coordinate group $F_{R(S_{i+1}, \dots, S_n)}$, where $X_{i+1}, \dots, X_n, A$ are viewed as constants.

Equivalently, in an NTQ system every equation $S_i(X_i) = 1$ has a solution in the generic point of the system $S_{i+1} = 1, \dots, S_n = 1$.

A TQ system

$$S_1(X_1, X_2, \ldots, X_n, A) = 1,$$

$$S_2(X_2, \ldots, X_n, A) = 1,$$

$$\ldots$$

$$S_n(X_n, A) = 1$$

is **non-degenerate (NTQ)** if for every i the equation $S_i(X_i, \ldots, X_n, A) = 1$ has a solution in the coordinate group $F_{R(S_{i+1},\ldots,S_n)}$, where $X_{i+1}, \ldots, X_n, A$ are viewed as constants.

Equivalently, in an NTQ system every equation $S_i(X_i) = 1$ has a solution in the generic point of the system $S_{i+1} = 1, \ldots, S_n = 1$.

Theorem C [With constants] *Let $\Gamma \in \mathcal{G}$. A finitely generated Γ-group H is Γ-universally equivalent to Γ if and only if H is embeddable into $\Gamma^{\mathbb{Z}[t]}$.*

Theorem D *Let $\Gamma \in \mathcal{G}$ and H a finitely generated group discriminated by Γ. Then H embeds into an NTQ extension of Γ.*

Theorem E *Let $\Gamma \in \mathcal{G}$ and Γ^* an NTQ extension of Γ. Then Γ^* embeds into a group $\Gamma(U, T)$ obtained from Γ by finitely many extensions of centarlizers.*

Theorem C [With constants] *Let $\Gamma \in \mathcal{G}$. A finitely generated Γ-group H is Γ-universally equivalent to Γ if and only if H is embeddable into $\Gamma^{\mathbb{Z}[t]}$.*

Theorem D *Let $\Gamma \in \mathcal{G}$ and H a finitely generated group discriminated by Γ. Then H embeds into an NTQ extension of Γ.*

Theorem E *Let $\Gamma \in \mathcal{G}$ and Γ^* an NTQ extension of Γ. Then Γ^* embeds into a group $\Gamma(U, T)$ obtained from Γ by finitely many extensions of centarlizers.*

Theorem C [With constants] *Let* $\Gamma \in \mathcal{G}$. *A finitely generated* Γ*-group* H *is* Γ*-universally equivalent to* Γ *if and only if* H *is embeddable into* $\Gamma^{\mathbb{Z}[t]}$.

Theorem D *Let* $\Gamma \in \mathcal{G}$ *and* H *a finitely generated group discriminated by* Γ. *Then* H *embeds into an NTQ extension of* Γ.

Theorem E *Let* $\Gamma \in \mathcal{G}$ *and* Γ^* *an NTQ extension of* Γ. *Then* Γ^* *embeds into a group* $\Gamma(U, T)$ *obtained from* Γ *by finitely many extensions of centarlizers.*

Theorem [Groves] Let $\Gamma \in \mathcal{G}$ and G a finitely generated freely indecomposable group with abelian JSJ decomposition D. Then there exists a finite collection $\{\eta_i : G \rightarrow L_i\}_{i=1}^n$ of proper quotients of G such that, for any homomorphism $h : G \rightarrow \Gamma$ which is not equivalent to an injective homomorphism there exists $h' : G \rightarrow \Gamma$ with $h \sim h'$ (the relation $\sim$ uses conjugation, canonical automorphisms corresponding to D and "bending moves"), $i \in \{1, \ldots, n\}$ and $h_i : L_i \rightarrow \Gamma$ so that $h' = \eta_i h_i$. The quotient groups L_i are fully residually Γ.

Proofs

According to the construction of Makanin-Razborov diagram the set $Hom(G, \Gamma)$ is divided into a finite number of families. Therefore one of these families contains a discriminating set of homomorphisms. Each family corresponds to a sequence of fully residually Γ groups

$$G = G_0, G_1, \ldots, G_n,$$

where G_{i+1} is a proper quotient of G_i and $\pi_i : G_i \to G_{i+1}$ is an epimorphism. By [KM], for a discriminating family π_i is a monomorphism for the following subgroups H in the JSJ decomposition D_i of G_i

1. H is a rigid subgroup in D_i;
2. H is an edge subgroup in D_i;
3. H is the subgroup of an abelian vertex groups A in D_i generated by the canonical images in A of the edge groups of the edges of D_i adjacent to A.

We need the following result. **Lemma** [KM]

(1) Let $H = A *_{\langle d \rangle} B$ and $\pi : H \to \bar{H}$ be a homomorphism such that the restrictions of π on A and B are injective. Put

$$H^* = \langle \bar{H}, y \mid [C_{\bar{H}}(\pi(d)), y] = 1 \rangle.$$

Then for every $u \in C_{H^*}((\pi(d)), u \notin C_{\bar{H}}(\pi(d))$, a map

$$\psi(x) = \begin{cases} \pi(x), & x \in A, \\ \pi(x)^u, & x \in B. \end{cases}$$

gives rise to a monomorphism $\psi : H \to H^*$.

(2) Let $H = \langle A, t \mid d^t = c \rangle$ and $\pi : H \to \bar{H}$ be a homomorphism such that the restriction of π on A is injective. Put

$$H^* = \langle \bar{H}, y \mid [C_{\bar{H}}(\pi(d)), y] = 1 \rangle.$$

Then for every $u \in C_{H^*}((\pi(d)), u \notin C_{\bar{H}}(\pi(d))$, a map

$$\psi(x) = \begin{cases} \pi(x), & x \in A, \\ u\pi(x), & x = t. \end{cases}$$

Proposition Let H be the fundamental group of the graph of groups with two vertices, v and w such that v is a QH vertex, $H_w = \Gamma \in \mathcal{G}$, and there is a retract from H onto Γ. Let S_Q be a punctured surface corresponding to a QH vertex group in this decomposition of H. One can find a retract δ and a finite set of simple closed curves on S_Q with the following properties:
1) each of them and all boundary elements of S_Q are mapped by δ into non-trivial elements in the iterated centralizer extension of $\Gamma * F$ (denote it H),
2) each connected component of the surface obtained by cutting S_Q along this family of s.c.c. has Euler characteristic -1,
3) the fundamental group of each of these connected components is mapped monomorphically into a 2-generated free subgroup of an iterated centralizer extension of Γ.