Lyndon's Completion for partially commutative groups, Part I

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History and motivation

Lyndon (1960) gave a description of the solution set of 1-variable equations over free groups in terms of *parametric words*.

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This led him to consider *groups with parametric exponents:*

Definition

A group G equipped with the action of the ring $\mathbb{Z}[t]$ subject to the following axioms:

1. $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;
2. $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;
3. $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;
4. if $[g, h] = 1$, then $(gh)^\alpha = g^\alpha h^\alpha$;

is called a $\mathbb{Z}[t]$-group.

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"Although we study only $Z[t_1, \ldots, t_n]$-groups, and, indeed, only free $Z[t_1, \ldots, t_n]$-groups, we believe the concept is of some more general interest."

B. H. Neumann, A. I. Malcev, R. Baer, P. G. Kontorovich, G. Baumslag

Motivation: Putting the pieces together

1. Definition of a group with parametric exponents;
2. Structure and properties: tensor completion, discrimination, minimality, etc;
3. Universal object for f.g. models of the universal theory of the free group or for f.g. f.r.f.=limit groups.
4. Applications

Tensor completion

Let G be a $\mathbb{Z}[t]$-group. Then a $\mathbb{Z}[t]$-group $G^{\mathbb{Z}[t]}$ is called the *tensor* $\mathbb{Z}[t]$*-completion* of the group G if $G^{\mathbb{Z}[t]}$ satises the following universal property:

1. there exists a $\mathbb{Z}[t]$-homomorphism $\lambda : G \to G^{\mathbb{Z}[t]}$ such that $\lambda(G)$ $\mathbb{Z}[t]$-generates $G^{\mathbb{Z}[t]}$;
2. for any $\mathbb{Z}[t]$-group H and a $\mathbb{Z}[t]$-homomorphism $\phi : G \to H$ there exists the unique $\mathbb{Z}[t]$-homomorphism $\psi : G^{\mathbb{Z}[t]} \to H$ such that $\lambda \circ \psi = \phi$.

Extensions of centralisers, CSA, BP

Definition

Let $w \in G$, then $H = \langle G, t \mid [t, C(w)] = 1 \rangle$ is called *extension of the centraliser of* w.

Definition

G is *CSA* if every maximal abelian subgroup of G is malnormal.

Definition

A group G satisfies the *separation condition*(=BP property) if for any positive integer k and any tuples $u = (u_1, \ldots, u_k)$ and $g = (g_1, \ldots, g_{k+1})$ of elements from G such that

$$[u_i^{g_{i+1}}, u_{i+1}] \neq 1 \text{ for } i = 1, \ldots, k-1,$$

there exists an integer $n = n(u, g)$ such that

$$g_1 u_1^{\alpha_1} g_2 \ldots g_k u_k^{\alpha_k} g_{k+1} \neq 1$$

for any integers $\alpha_1, \ldots, \alpha_k \geq n$.

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     $$[u_i^{g_{i+1}}, u_{i+1}] \neq 1 \quad \text{for} \quad i = 1, \dots, k-1.$$

     Use some ϕ to map these non-trivially. Since G is BP, there is an n s. t.

     $$\phi(g_1 u_1^{\alpha_1} g_2 \dots g_k u_k^{\alpha_k} g_{k+1}) \neq 1$$

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Hence, we have $F \hookrightarrow H_1 \hookrightarrow H_2 \hookrightarrow \dots$ and $\bigcup H_i = \widetilde{F}$.

Structure of $\widetilde{F}$: free groups

- The property of being CSA is invariant w.r.t. extensions of centralisers and transfinite induction. Hence $\widetilde{F}$ is CSA.
- The centraliser of an element g from $\widetilde{F}$ is a 1-generated $\mathbb{Z}[t]$-module.
- Using properties of the action and the CSA property, one extends the action to $\widetilde{F}$.
- The universal property of the centralisers extends to $\widetilde{F}$.
- One can conclude that $\widetilde{F}$ is the tensor completion of the group F.

Universal object for limit groups

Theorem (Kharlampovich, Miasnikov)

A finitely generated group G is a limit group if and only if G is a subgroup of $\widetilde{F}$.

Proof.

- Understanding radicals of quadratic equations;
- Makanin-Razborov machinery.

Applications

- Cyclic splittings of limit groups;
- Limit groups are finitely presented;
- Hierarchy on limit groups;
- Infinite words and algorithmic problems for limit groups.

Partially commutative groups

Let Γ be a (finite, undirected, simple) graph, $A = V(\Gamma) = \{a_1, \ldots, a_n\}$. Let

$$R = \{[a_i, a_j] \mid a_i, a_j \in A \text{ are connected in } \Gamma\}.$$

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- Let w be a c. r. word in G. Set $alph(w)$ to be the set of all symbols which occur in w.
- Consider $\Delta(w) = \Delta(alph(w))$
- If $\Delta(w)$ is not connected then w splits into a product of commuting words $\left\{w^{(j)} | j \in J\right\}$.
- $w^{(j)}$ involves the letters from j-th component only;
- $\left[w^{(j_1)}, w^{(j_2)}\right] = 1$;

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Definition

If w is cyclically reduced and $\Delta(w)$ is connected, w is called a block. Any element conjugate to w is also called a block.

Centralisers in partially commutative groups

Theorem (H. Servatius 1987, G. Duchamp and D. Krob 1993)
Let w be a cyclically reduced element of G, $w = w^{(1)} \cdots w^{(k)}$ be its block decomposition.

$$C(w) = \langle \sqrt{w^{(1)}} \rangle \times \cdots \times \langle \sqrt{w^{(k)}} \rangle \times A(w),$$

where $A(w)$ is a subgroup of G generated by all words that commute with w and do not occur in w.

Equations in one variable revisited

$$E:\ [x,w]=1, \qquad V(E)=C(w)=\langle\sqrt{w^{(1)}}\rangle\times\cdots\times\langle\sqrt{w^{(k)}}\rangle\times A(w),$$

BP, CSA and partially commutative groups

- Partially commutative groups are *far* from being CSA.
- They are not BP.

Theorem (V. Blatherwick)

A partially commutative group is BP if and only if its commutation graph does not contain n-cycles for $n \geq 4$.

Age of pessimism

Go to the blackboard and give 2 examples: BP and discrimination.