

# Lyndon's Completion for partially commutative groups, Part I

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## History and motivation

Lyndon (1960) gave a description of the solution set of 1-variable equations over free groups in terms of *parametric words*.

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This led him to consider *groups with parametric exponents*:

## Definition

A group  $G$  equipped with the action of the ring  $\mathbb{Z}[t]$  subject to the following axioms:

1.  $g^1 = g, g^0 = 1, 1^\alpha = 1$ ;
2.  $g^{\alpha+\beta} = g^\alpha \cdot g^\beta, g^{\alpha\beta} = (g^\alpha)^\beta$ ;
3.  $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ ;
4. if  $[g, h] = 1$ , then  $(gh)^\alpha = g^\alpha h^\alpha$ ;

is called a  $\mathbb{Z}[t]$ -group.

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### Theorem

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*"Although we study only  $Z[t_1, \dots, t_n]$ -groups, and, indeed, only free  $Z[t_1, \dots, t_n]$ -groups, we believe the concept is of some more general interest."*

B. H. Neumann, A. I. Malcev, R. Baer, P. G. Kontorovich, G. Baumslag

# Motivation: Putting the pieces together

1. Definition of a group with parametric exponents;
2. Structure and properties: tensor completion, discrimination, minimality, etc;
3. Universal object for f.g. models of the universal theory of the free group or for f.g. f.r.f.=limit groups.
4. Applications

# Tensor completion

Let  $G$  be a  $\mathbb{Z}[t]$ -group. Then a  $\mathbb{Z}[t]$ -group  $G^{\mathbb{Z}[t]}$  is called the *tensor  $\mathbb{Z}[t]$ -completion* of the group  $G$  if  $G^{\mathbb{Z}[t]}$  satisfies the following universal property:

1. there exists a  $\mathbb{Z}[t]$ -homomorphism  $\lambda : G \rightarrow G^{\mathbb{Z}[t]}$  such that  $\lambda(G)$   $\mathbb{Z}[t]$ -generates  $G^{\mathbb{Z}[t]}$ ;
2. for any  $\mathbb{Z}[t]$ -group  $H$  and a  $\mathbb{Z}[t]$ -homomorphism  $\phi : G \rightarrow H$  there exists the unique  $\mathbb{Z}[t]$ -homomorphism  $\psi : G^{\mathbb{Z}[t]} \rightarrow H$  such that  $\lambda \circ \psi = \phi$ .



# Extensions of centralisers, CSA, BP

## Definition

Let  $w \in G$ , then  $H = \langle G, t \mid [t, C(w)] = 1 \rangle$  is called *extension of the centraliser of  $w$* .

## Definition

$G$  is CSA if every maximal abelian subgroup of  $G$  is malnormal.

## Definition

A group  $G$  satisfies the *separation condition* (=BP property) if for any positive integer  $k$  and any tuples  $u = (u_1, \dots, u_k)$  and  $g = (g_1, \dots, g_{k+1})$  of elements from  $G$  such that

$$[u_i^{g_{i+1}}, u_{i+1}] \neq 1 \text{ for } i = 1, \dots, k-1,$$

there exists an integer  $n = n(u, g)$  such that

$$g_1 u_1^{\alpha_1} g_2 \dots g_k u_k^{\alpha_k} g_{k+1} \neq 1$$

for any integers  $\alpha_1, \dots, \alpha_k \geq n$ .

# Construction of the completion

Let  $G$  be CSA and BP and let  $H$  be obtained by extending the centraliser of  $w \in G$ . Then

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$$[u_i^{g_i+1}, u_{i+1}] \neq 1 \text{ for } i = 1, \dots, k-1.$$

Use some  $\phi$  to map these non-trivially. Since  $G$  is BP, there is an  $n$  s. t.

$$\phi(g_1 u_1^{\alpha_1} g_2 \dots g_k u_k^{\alpha_k} g_{k+1}) \neq 1$$

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Hence, we have  $F \hookrightarrow H_1 \hookrightarrow H_2 \hookrightarrow \dots$  and  $\bigcup H_i = \tilde{F}$ .

## Structure of $\tilde{F}$ : free groups

- ▶ The property of being CSA is invariant w.r.t. extensions of centralisers and transfinite induction. Hence  $\tilde{F}$  is CSA.
- ▶ The centraliser of an element  $g$  from  $\tilde{F}$  is a 1-generated  $\mathbb{Z}[t]$ -module.
- ▶ Using properties of the action and the CSA property, one extends the action to  $\tilde{F}$ .
- ▶ The universal property of the centralisers extends to  $\tilde{F}$ .
- ▶ One can conclude that  $\tilde{F}$  is the tensor completion of the group  $F$ .



# Universal object for limit groups

## Theorem (Kharlampovich, Miasnikov)

*A finitely generated group  $G$  is a limit group if and only if  $G$  is a subgroup of  $\tilde{F}$ .*

Proof.

- ▶ Understanding radicals of quadratic equations;
- ▶ Makanin-Razborov machinery.

# Applications

- ▶ Cyclic splittings of limit groups;
- ▶ Limit groups are finitely presented;
- ▶ Hierarchy on limit groups;
- ▶ Infinite words and algorithmic problems for limit groups.

## Partially commutative groups

Let  $\Gamma$  be a (finite, undirected, simple) graph,

$A = V(\Gamma) = \{a_1, \dots, a_n\}$ . Let

$$R = \{[a_i, a_j] \mid a_i, a_j \in A \text{ are connected in } \Gamma\}.$$

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- ▶ Let  $w$  be a c. r. word in  $G$ . Set  $\text{alph}(w)$  to be the set of all symbols which occur in  $w$ .
- ▶ Consider  $\Delta(w) = \Delta(\text{alph}(w))$
- ▶ If  $\Delta(w)$  is not connected then  $w$  splits into a product of commuting words  $\{w^{(j)} \mid j \in J\}$ .
- ▶  $w^{(j)}$  involves the letters from  $j$ -th component only;
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## Definition

If  $w$  is cyclically reduced and  $\Delta(w)$  is connected,  $w$  is called a block. Any element conjugate to  $w$  is also called a block.

# Centralisers in partially commutative groups

Theorem (H. Servatius 1987, G. Duchamp and D. Krob 1993)

Let  $w$  be a cyclically reduced element of  $G$ ,  $w = w^{(1)} \dots w^{(k)}$  be its block decomposition.

$$C(w) = \langle \sqrt{w^{(1)}} \rangle \times \dots \times \langle \sqrt{w^{(k)}} \rangle \times A(w),$$

where  $A(w)$  is a subgroup of  $G$  generated by all words that commute with  $w$  and do not occur in  $w$ .

## Equations in one variable revisited

$$E : [x, w] = 1, \quad V(E) = C(w) = \langle \sqrt{w^{(1)}} \rangle \times \cdots \times \langle \sqrt{w^{(k)}} \rangle \times A(w),$$



# BP, CSA and partially commutative groups

- ▶ Partially commutative groups are *far* from being CSA.
- ▶ They are not BP.

## Theorem (V. Blatherwick)

*A partially commutative group is BP if and only if its commutation graph does not contain  $n$ -cycles for  $n \geq 4$ .*

# Age of pessimism

Go to the blackboard and give 2 examples: BP and discrimination.