On systems of equations with harmonic polynomials

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Notation

- M is a compact connected homogeneous Riemannian manifold, $M = G/H$, where H is the stable subgroup of a base point o.
- The isotropy representation of H in $\mathfrak{t} = T_o M$ is assumed to be irreducible. Then M admits the unique up to a scaling factor G-invariant Riemannian metric.
- Let $\langle\ ,\ \rangle$, Δ, and $\mathfrak{h}^k$ be the corresponding inner product in $\mathfrak{t}$, Laplace–Beltrami operator, and Hausdorff measure of dimension k, respectively.
- $m = \dim M$.

- Let $\mathcal{E}$ be a G- and Δ-invariant subspace of $C^\infty(M)$, and $\mathcal{S}$ be the unit sphere in $\mathcal{E}$ with respect to the norm of $L^2(M)$.
- We write $\int_M f(u)\,du$, $\int_G f(g)\,dg$, etc., for integration over the invariant probability measures on M, G, and other compact homogeneous spaces.
- The mean value (expectation) of a function f on $\mathcal{S}$ is defined as

$$\mathrm{M}(f) = \int_{\mathcal{S}} f(u)\,du.$$

- For a real function u on M and $t \in \mathbb{R}$, set

$$L_u^t = \{x \in M : \ u(x) = t\},$$
$$U_u^t = \{x \in M : \ u(x) \geq t\}.$$

Leading examples

- Let M be the unit sphere S^m in $\mathbb{R}^{m+1}$, $G = \mathrm{SO}(m+1)$. Then

$$\varpi = \varpi_m = \mathfrak{h}^m(S^m) = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)},$$
$$L_2(S^m) = \sum_{j=0}^{\infty} \oplus \mathcal{H}_j^m,$$

  where $\mathcal{H}_n^m$ is the space of harmonic homogeneous polynomials of degree n on $\mathbb{R}^{m+1}$, as well as their restrictions to S^m. Furthermore, $\Delta u + n(n+m-1)u = 0$ for all $u \in \mathcal{H}_n^m$. Functions in $\mathcal{H}_n^m$ are called *spherical harmonics*. The space $\mathcal{H}_n^m$ is G-invariant and irreducible; $\mathcal{E}$ may be a finite sum of these spaces.
- A simple Lie group G is an isotropy irreducible homogeneous space of $G \times G$ acting on G by left and right shifts.

Questions

- Let $\mathcal{E}_j$ and $\mathcal{S}_j$ be as $\mathcal{E}$ and $\mathcal{S}$ above, $\mathcal{E}_j$ pairwise orthogonal, $j = 1, \dots, k$.
- For $\mathbf{t} = (t_1, \dots, t_k)$, $\mathbf{u} = (u_1, \dots, u_k)$, where $t_j \in \mathbb{R}$ and $u_j \in \mathcal{S}_j$, set

$$U_{\mathbf{u}}^{\mathbf{t}} = \bigcap_{j=1}^{k} U_{u_j}^{t_j}, \quad L_{\mathbf{u}}^{\mathbf{t}} = \bigcap_{j=1}^{k} L_{u_j}^{t_j}.$$

Then $L_{\mathbf{u}}^{\mathbf{t}}$ is the set of solutions to the system of equations

$$\mathbf{u}(x) = \mathbf{t}.$$

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There are natural questions:

- how massive can the set $L_{\mathbf{u}}^{\mathbf{t}}$ be?
- what is the mean size of it?
- what can one say about the geometry of this set?

How to measure it?

The *Hausdorff measure* $\mathfrak{h}^s$ of dimension s is defined in two steps:

1) Let $\delta > 0$ and

$$\mathfrak{h}^s_\delta(E) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s+1}{2}\right)} \inf \left\{ \sum \left(\frac{\operatorname{diam} C}{2}\right)^s : E \subseteq \bigcup C, \quad \operatorname{diam} C < \delta \right\};$$

2) set $\mathfrak{h}^s(E) = \sup_{\delta > 0} \mathfrak{h}^s_\delta(E)$.

The measure $\mathfrak{h}^0$ is the counting function (i.e., $\mathfrak{h}^0(E) = \operatorname{card}(E)$).

Embedding $M \to \mathcal{E}$

There is a natural mapping $\iota : M \to \mathcal{S}$,

$$\iota(a) = \frac{\phi_a}{|\phi_a|},$$

where $\phi_a \in \mathcal{E}$ realizes the evaluation functional at a:

$$\int_M u(p)\phi_a(p)\, dp = u(a) \quad \text{for all} \quad u \in \mathcal{E}.$$

Locally, the mapping ι is an embedding and a metric homothety with the coefficient

$$s = \frac{|d_a\iota(v)|}{|v|},$$

where the right-hand side is independent of $a \in M$ and $v \in T_aM \setminus \{0\}$.

The following formula is a very useful tool ("k-rectifiable" means "Lipschitz image of a bounded subset of $\mathbb{R}^k$").

Theorem (Federer)

Let $A, B \subseteq S^d$ be compact, A be k-rectifiable, and B be l-rectifiable. Set $r = k + l - d$. Suppose $r \geq 0$. Then

$$\int_{\mathrm{O}(d+1)} \mathfrak{h}^r(A \cap gB)\, dg = K \mathfrak{h}^k(A) \mathfrak{h}^l(B),$$

where $K = \frac{\varpi_r}{\varpi_k \varpi_l}$.

Lemma

Let $|t| \leq 1$ *and* $X \subseteq M$ *be* $(r+1)$*-rectifiable, where* $r \leq m-1$*. Then*

$$\int_{\mathcal{S}} \mathfrak{h}^r(L_u^{ct} \cap X)\, du = \frac{\varpi_r}{\varpi_{r+1}} s \left(1 - t^2\right)^{\frac{d-1}{2}} \mathfrak{h}^{r+1}(X) \qquad (1)$$

If ι is one-to-one on X, then $\mathfrak{h}^r(\iota(X)) = s^r \mathfrak{h}^r(X)$ since ι is a metric homothety locally.

Mean Hausdorff measure of level sets

If $\Delta u = -\lambda u$ for all $u \in \mathcal{E}$, then $s = \sqrt{\frac{\lambda}{m}}$. If $\mathcal{E} = \sum_{i=1}^{k} \oplus \mathcal{E}^i$, where $\Delta u = -\lambda_i u$ for all $u \in \mathcal{E}^i$ and $\mathcal{E}^i$ are pairwise orthogonal, then

$$s = \sqrt{\frac{1}{m}(\alpha_1 \lambda_1 + \cdots + \alpha_k \lambda_k)},$$

where $\alpha_i = \frac{\dim \mathcal{E}^i}{\dim \mathcal{E}}$. Set

$$c = |\phi_o|,$$
$$d = \dim \mathcal{S} = \dim \mathcal{E} - 1.$$

Theorem

For any $t \in [-c, c]$, the following equalities hold:

$$\mathrm{M}\left(\mathfrak{h}^{m-1}(L_u^t)\right) = \varpi \frac{\omega_{m-1}}{\omega_m} s \left(1 - \frac{t^2}{c^2}\right)^{\frac{d-1}{2}};$$

$$\mathrm{M}(\mathfrak{h}^m(U_u^t)) = \varpi \frac{\varpi_{d-1}}{\varpi_d} \int_{\frac{t}{c}}^{1} \left(1 - \tau^2\right)^{\frac{d}{2}-1} d\tau.$$

If $M = S^m$, $\mathcal{E} = \mathcal{H}_n^m$, and $t = 0$, then we get the mean value of the Hausdorff measures of nodal sets of spherical harmonics; it is equal to $\frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \sqrt{\frac{n(n+m-1)}{m}}$. By different but similar methods, it was found by Berard (1984) and Neuheisel (2000).

Higher moments and variance of $\mathfrak{h}^m(U_u^t)$

The kth moment of a function f on $\mathcal{S}$ is defined as

$$\mathrm{M}_k(f(u)) = \mathrm{M}(f(u)^k).$$

For $\tau \in [-1, 1]$ and $a \in \mathcal{S}$, set

$$\mathcal{U}_a^\tau = \{x \in \mathcal{S} : \langle x, a\rangle \geq \tau\}.$$

Then, for each $t \in [-c, c]$,

$$\mathrm{M}_k\left(\mathfrak{h}^m(U_u^t)\right) = \frac{\varpi^k}{\varpi_d} \int_{M^k} \mathfrak{h}^d\left(\mathcal{U}_{\iota(p_1)}^{\frac{t}{c}} \cap \ldots \cap \mathcal{U}_{\iota(p_k)}^{\frac{t}{c}}\right) dp_1 \ldots dp_k$$

This implies a formula for the variance:

$$\operatorname{var}\left(\mathfrak{h}^m(U_u^0)\right) = \frac{\varpi^2}{2\pi}\int_M \arcsin\frac{\phi_o(p)}{c^2}\,dp.$$

In particular, for S^2 and $\mathcal{H}_n^2$ we get

$$8\pi\int_{-1}^{1}\arcsin P_n(x)\,dx,$$

where P_n is the nth Legendre polynomial.

Moments for intersections of level sets

Let $\mathcal{E}_j$, $\mathcal{S}_j$, where $j = 1, \dots, l$ and $l \leq m$, be as $\mathcal{E}$, $\mathcal{S}$ above, as well as s_j, c_j, d_j. Set $\mathbf{t} = (t_1, \dots, t_l)$, $\mathbf{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_k$, $\mathbf{u} = (u_1, \dots, u_l) \in \mathbf{S}$, $L_{\mathbf{u}}^{\mathbf{t}} = L_{u_1}^{t_1} \cap \cdots \cap L_{u_l}^{t_l}$. Then

$$\mathrm{M}\left(\mathfrak{h}^{m-l}\left(L_{\mathbf{u}}^{\mathbf{t}}\right)\right) = \varpi \frac{\omega_{m-l}}{\omega_m} \prod_{j=1}^{l} s_j \left(1 - \frac{t_j^2}{c_j^2}\right)^{\frac{d_j - 1}{2}} .$$

For $U_{\mathbf{u}}^{\mathbf{t}}$, we have

$$\mathrm{M}_k\left(\mathfrak{h}^{m}\left(U_{\mathbf{u}}^{\mathbf{t}}\right)\right) = \varpi^{k(1-l)} \prod_{j=1}^{l} \mathrm{M}_k\left(\mathfrak{h}^{m}\left(U_{u_j}^{t_j}\right)\right)$$

Some particular cases

- If $l = m$ and $\mathbf{t} = 0$, then we get the mean number of common zeroes:

$$\mathrm{M}\left(\mathfrak{h}^0\left(L^0_{\mathbf{u}}\right)\right) = 2\frac{\varpi}{\omega_m} s_1 \dots s_m$$

since $\omega_0 = 2$; in particular, for S^m and $\mathcal{E}_j = \mathcal{H}^m_j$ we have $2m^{-\frac{m}{2}}\sqrt{\lambda_{j_1}\dots\lambda_{j_m}}$, where $\lambda_j = j(j+m-1)$.

- Let $M = S^2$, $\mathcal{P}_n$ be the space of all homogeneous polynomials of degree n (restricted to S^2), and $\mathcal{H}_n \subset \mathcal{P}_n$ be its subspace of harmonic polynomials. Then

$$k = 1: \quad \mathrm{M}(\mathcal{P}_n) \approx \pi n, \quad \mathrm{M}(\mathcal{H}_n) = \pi\sqrt{2n(n+1)};$$
$$k = 2: \quad \mathrm{M}(\mathcal{P}_n, \mathcal{P}_n) \approx \tfrac{n^2}{2}, \quad \mathrm{M}(\mathcal{H}_n, \mathcal{H}_n) = n(n+1),$$

where M denotes the mean value of Hausdorff measures of nodal sets of intersection of k functions in the corresponding spaces.

How many common zeroes may have two spherical harmonics on S^2?

Let $\mathcal{E} = \mathcal{H}_n^2$. Then $\lambda = n(n+1)$.

- The set $N_u \cap N_v$ may be infinite. For example, polynomials of the type $xP(y,z)$, where $\Delta P = 0$, are harmonic.
- If $N_u \cap N_v$ is finite, then $\mathrm{card}(N_u \cap N_v) \leq 2n^2$.
- Conjecture: $\mathrm{card}(N_u \cap N_v) \geq 2n$ (supported by computer experiments and partial results).

Examples

- Let ϕ_a be the zonal harmonic: $\phi_a(x) = c_n P_n(\langle a, x\rangle)$, where $a, x \in S^2$, c_n is a normalizing constant, and P_n is the nth Legendre polynomial: $P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$.
- If $a, b \in S^2$ are sufficiently close, then

$$\mathrm{card}(N_{\phi_a} \cap N_{\phi_b}) = 2n.$$

- Let $S_0 = \{z \in \mathbb{C}^3 : \langle z, z\rangle = 0\}$, where $\langle\ ,\ \rangle$ is the bilinear extension of the standard inner product in $\mathbb{R}^3$ onto $\mathbb{C}^3$. Set $\psi_a(z) = \mathrm{Re}\,\langle a, z\rangle^n$. If $a \in S_0$, then $\psi_a \in \mathcal{H}_n$.
  For generic $a, b \in S_0$

$$\mathrm{card}(N_{\psi_a} \cap N_{\psi_b}) = 2n^2.$$

Remarks on critical points

- If the number of critical points for $u \in \mathcal{H}_n$ is finite, then it does not exceed $2n^2$.
- The bound is not sharp. I do not know an example of u which has more than $2(n^2 - n + 1)$ critical points.
- The configuration of the critical points is always degenerate.
- If $p \in \mathcal{H}_3$ is symmetric and is not zonal, then it has 14 critical points lying on circles $x = y$, $y = z$, $z = x$ $(14 = 2(3^2 - 3 + 1))$.

A determinant formula

Any polynomial $p \in \mathcal{H}_n^{\mathbb{C}}$ can be extended holomorphically onto $\mathbb{C}^3$. The set $p^{-1}(0) \cap S_0$ is the union of complex lines $\mathbb{C}a_1, \dots, \mathbb{C}a_{2n}$. Some of them may coincide. If they are distinct, then

$$p(x)p(y) = C \det \begin{pmatrix} \langle a_1, a_1 \rangle^n & \dots & \langle a_1, a_{2n} \rangle^n & \langle a_1, y \rangle^n \\ \vdots & \ddots & \vdots & \vdots \\ \langle a_{2n}, a_1 \rangle^n & \dots & \langle a_{2n}, a_{2n} \rangle^n & \langle a_{2n}, y \rangle^n \\ \langle x, a_1 \rangle^n & \dots & \langle x, a_{2n} \rangle^n & P_n(x, y) \end{pmatrix},$$

where $C \neq 0$ is constant and $P_n(x, y)$ is the unique extension of $P_n(\langle x, y \rangle)$ from S^2 onto $\mathbb{C}^3$ which belongs to $\mathcal{H}_n^{\mathbb{C}}$ on each variable.

- Brüning (1978): the lower bound $c\sqrt{\lambda}$ for the length of a nodal set of a λ-eigenfunction on a surface.
- Yau's conjecture (1982):

$$c\sqrt{\lambda} < \mathfrak{h}^{m-1}(N_u) < C\sqrt{\lambda},$$

  where u is an eigenfunction, $m = \dim M$.
- Donnelly and Fefferman (1988): the case of real analytic M.
- Savo (2001): lower bound with $c = \frac{1}{11}\,\mathrm{Area}(M)$ for surfaces (for all sufficiently large λ in any surface and for all λ if the curvature is nonnegative).
- Mangoubi (2005, 2007): upper and lower bounds for the inner radius; for surfaces, they are of the type $\frac{c}{\sqrt{\lambda}}$.

Notation

- $\operatorname{inr}(D) = \sup_{p \in D} \operatorname{dist}(p, \partial D)$: the inner radius of D;
- $j_0 \approx 2.4048$: the least positive root of the Bessel function J_0;
- $\cos\theta_n$: the greatest root of the Legendre polynomial P_n, $\theta_n \approx \frac{j_0}{n+\frac{1}{2}}$;
- $\lambda_n = n(n+1)$.

Estimates of nodal length and inner radius for S^2

- In the case of S^2, the following inequalities hold:

$$\frac{2\pi}{j_0}\left(n+\frac{1}{2}\right) < \mathfrak{h}^1(N_u) \le 2\pi n;$$

$$\arcsin\frac{1}{n} \le \operatorname{inr}\left(\mathbb{S}^2 \setminus N_u\right) \le \theta_n < \frac{j_0}{n+\frac{1}{2}}.$$

- The upper bounds $2\pi n$ and θ_n are attained (for functions $\mathrm{Re}(x+yi)^n$ and ϕ_a, $a \in S^2$, respectively; estimates for the inner radius are given for the inner metric in S^2).
- The lower bound above is greater than $\frac{1}{11}\mathrm{Area}(M)\sqrt{\lambda}$:

$$\frac{4\pi}{11}\sqrt{n(n+1)} < \frac{2\pi}{j_0}\left(n+\frac{1}{2}\right),$$

where $\frac{4\pi}{11} \approx 1.4248$, $\frac{2\pi}{j_0} \approx 2.6127$. Nevertheless, it seems to be far from being sharp.

Theorem

- (1) *Suppose* $H^1(M) = 0$. *Then for any* $\lambda \neq 0$ *and each pair* $u, v \in \mathcal{E}_\lambda$ *there exists* $p \in M$ *such that* $u(p) = v(p) = 0$.

  (2) *If* M *is a homogeneous space of a compact Lie group* G *of isometries, then the converse is true:* $H^1(M) \neq 0$ *implies the existence of* $\lambda \neq 0$ *and a pair* $u, v \in \mathcal{E}_\lambda$ *without common zeroes.*

  - The simplest example for (2): $M = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $u(t) = \cos t$, $v(t) = \sin t$.
  - Moreover, (2) is an easy consequence of this example since $H^1(M) \neq 0$ implies that M admits a G_0-invariant fibration over $\mathbb{T}$, where G_0 is the identity component of G.

A sketch of the proof of (1)

- If $N_u \cap N_v = \emptyset$, then the family of the nodal domains for u and v is a covering of M. Let $\mathfrak{U}, \mathfrak{V}$ be the families of the nodal domains for u, v, respectively.
- Let $u, v \in \mathcal{E}_\lambda$ and denote by $\Gamma_{u,v}$ the graph whose vertices are the nodal domains and edges join domains with nonempty intersection.
- The graph $\Gamma_{u,v}$ has the properties:
  (a) each its edge joins $\mathfrak{U}$ and $\mathfrak{V}$;
  (b) each vertex is common for at least two edges.

  The property (b) holds since

  (A) If $U \in \mathfrak{U}$, $V \in \mathfrak{V}$ and $U \subseteq V$, then $u = cv$ for some $c \in \mathbb{R}$;
  (B) u (or v) takes values of different signs near its nodal set.

- The following corollary answers to a question of a paper by Galindo, de la Harpe, and Vust in affirmative.

Corollary

Let G be a compact connected irreducible linear group acting in a complex linear space V such that $\dim V > 1$ and let H be a hyperplane in V. For each $v \in V$, $Gv \cap H \neq \emptyset$.