A survey of partially commutative groups

Andrew Duncan

August 17th, 2009

Outline

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1. Introduction
2. Subgroups
3. Embeddings
4. Geometric techniques
5. Combinatorial techniques
6. Automorphisms

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Partially Commutative Groups

Γ a graph, with vertices $X = \{x_1, \ldots, x_n\}$ and edges E a set of 2-subsets of X.

The *partially commutative group* $G = G(\Gamma)$ is

$$\langle X \mid [x, y] = 1, \forall \{x, y\} \in E \rangle.$$

(semi-free, graph, right-angled, trace, locally free)
e.g.
free groups: Γ the null graph
free Abelian groups: Γ the complete graph

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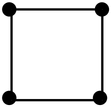

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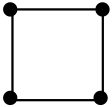

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Uncles, Aunts, Sisters and Cousins

- Artin Groups: $\langle X | R \rangle$, relators R of form $(x_i x_j)^{m_{ij}} = (x_j x_i)^{m_{ij}}$, m_{ij} a non-negative integer.
- Coxeter Groups: as Artin groups with all relators $x_i^2 = 1$ added.
- Partially commutative Coxeter groups: $m_{ij} = 0$ or 1.
- Partially commutative products $G_1 * \ldots * G_n / N(R)$; where $R = \{[g_i, g_j] = 1 \iff \{x_i, x_j\} \in E, g_i \in G_i, g_j \in G_j\}$.
- Trace monoids: $M(\Gamma) = \langle X \,|\, xy = yx, \iff \{x, y\} \in E \rangle$
- Partially commutative associative algebras: $K\langle X \rangle / I(R)$ ...
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Normal forms

- Good normal forms (Baudisch, Acta Math. Hung. '77):

  non-commutation graph Δ: $V(\Delta) = X$,
  $e \in E(\Delta) \iff e \notin E(\Gamma)$.

  In $\mathbb{F}_2 \times \mathbb{F}_2$: $w = a^2 b^{-1} cab$

  $\Delta(w)$ = full subgraph of Δ generated by letters of w:

  Gathers w into "blocks": $w = (a^2 ca)(b^2)$, each block having letters from a component of $\Delta(w)$.
- Unique normal form (after X ordered).
- Rewrite to n.f. using only cancellation and commutation (no length increases) so solvable word problem. Group elements have well-defined *length*.
- Many other normal forms (e.g. Duchamp & Krob; Diekert; Kazachkov, Esyp and Remeslennikov).

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Torsion, roots, 2-generators

- *Torsion and roots*: G is torsion-free and has unique roots: for all non-trivial $g \in G$ there exists unique $r \in G$ and $n > 0$ such that $g = r^n$ and if $g = h^m$ then $h = r^l$, for some l.
- Any two elements of G which do not commute generate a non-Abelian free group (Baudisch, Acta. Math. Hung. '81).
- This result does not carry over to 3 elements.
- *Stallings' example (Sem. Bourbaki, '76)*: $G = \mathbb{F}(a, c) \times \mathbb{F}(b, d)$ and

$$\pi : G \to \mathbb{Z}, \quad a\pi = b\pi = 0,\ c\pi = 1,\ d\pi = -1.$$

$\ker(\pi) = N = \langle a, b, cd \rangle$ is not finitely presentable ($H_2(N; \mathbb{Q})$ is not f.g. over $\mathbb{Q}$)

so G is not coherent (and not $\pi_1(M)$ for any 3-manifold M).

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$$\pi : G \to \mathbb{Z}, \quad a\pi = b\pi = 0,\ c\pi = 1,\ d\pi = -1.$$

  $\ker(\pi) = N = \langle a, b, cd \rangle$ is not finitely presentable ($H_2(N; \mathbb{Q})$ is not f.g. over $\mathbb{Q}$)

  so G is not coherent (and not $\pi_1(M)$ for any 3-manifold M).

Parabolic subgroups (Baudisch, '77)

- *Parabolic subgroups*: If $Y \subseteq X$ and Γ_Y is the full subgraph of Γ generated by Y then $\langle Y \rangle = G(\Gamma_Y)$, the partially commutative group of Γ_Y.
- *Stars*: Let $x^\perp = \{y \in X \mid [x, y] = 1\}$.
- *Letters*: For $w \in G$ write $V(w)$ for the letters of X occuring in a normal form of w.
- *Links*: For $w \in G$ set

$$L(w) = \left(\bigcap_{x \in V(w)} x^\perp \right) \setminus V(w).$$

  e.g. $L(x) = x^\perp \setminus x$.
- *Centraliser*: If w is cyclically reduced (of minimal length in it's conjugacy class) then its centraliser is

$$C(w) = \langle u_1 \rangle \times \cdots \times \langle u_r \rangle \times \langle L(w) \rangle,$$

  where $w = u_1^{a_1} \cdots u_r^{a_r}$, in n.f. and u_i is a root element (u_i is a block, not a letter).

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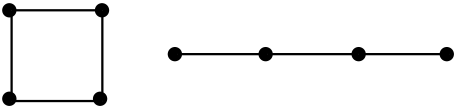
When G is coherent

- *Coherence, Droms, J. Alg. '87*: Every finitely generated subgroup of G is a graph group if and only if Γ has no full subgraph isomorphic to either of



- Such graphs are *chordal*; every connected chordal graph contains a central vertex.
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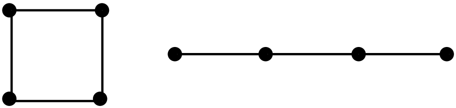

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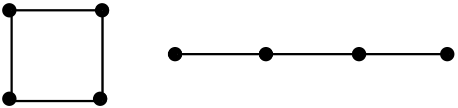

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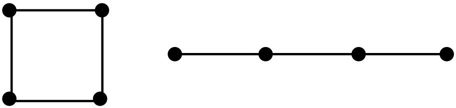

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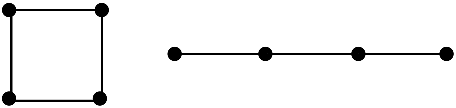

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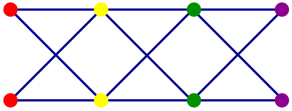
Questions

- In general which subgroups are finitely presented? How can we recognise them?
- Which subgroups are finitely generated?
- Which have solvable membership problem?
- Can we decide malnormality of subgroups of pc groups of chordal graphs?

Partial answers to some of these questions: e.g. Baumslag & Roseblade, Bogopolski & Ventura, Hsu & Wise.

Embeddings of and into...

- $G(\Gamma)$ is a finite index subgroup of $C(\Gamma)$, the p.c. Coxeter group with graph 2Γ:



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- G is linear (Humphries) so residually finite.
- G is residually nilpotent Duchamp & Krob, '92:by embedding G in the Magnus subgroup of $K(\Gamma)$, where K a field of charecteristic 0, $K(\Gamma) = K[X]/I(\Gamma)$ and $I(\Gamma)$ is generated by $(w - u)$ such that $w \equiv u \in M(\Gamma)$.
- G embeds into a pure braid group PB_m, for sufficiently large m (Crisp & Wiest).
- Graph braid groups embed into $G(\Gamma)$ for some G.
- $\pi_1(S)$ is isomorphic to a subgroup of $G(C_5)$, for any closed orientable surface S (Röver, '04).Most surface groups embed in some $G(\Gamma)$ (Crisp, Wiest, '03).

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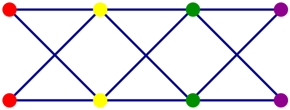

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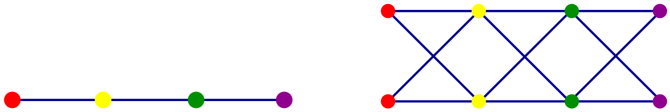

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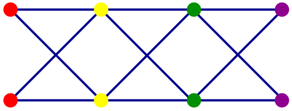

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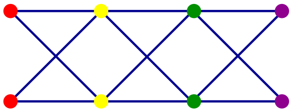

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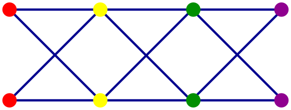

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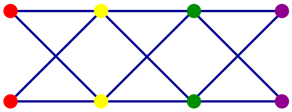

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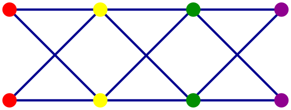

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The Salvetti Complex

The *flag complex* Γ_f of Γ: add an n-simplex $\sigma\{v_0, \ldots, v_n\}$ whenever $v_1, \ldots, v_n$ spans a complete subgraph of Γ.

Γ Γ_f

Construct a $K(G, 1)$ space K:

1 vertex $*$

n edges attached to $*$, oriented and labelled $x_1, \ldots, x_n$

one 2-cube for each edge of Γ;

for edge $\{x, y\}$ attach a square with boundary label $[x, y]$ to K^1

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Theorem (Davis & Charney, Ann. of Math. Stud. '95)

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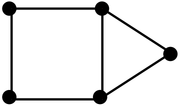

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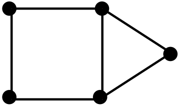

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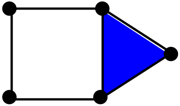

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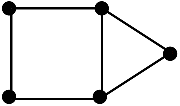

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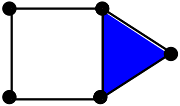

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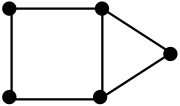

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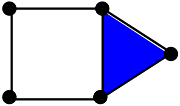

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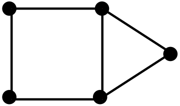

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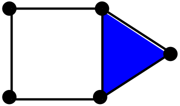

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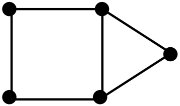
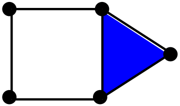
Γ  Γ_f 

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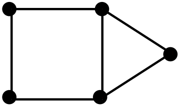

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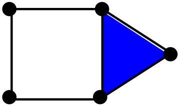

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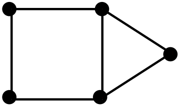
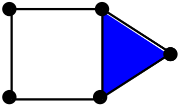
Γ  Γ_f 

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A group has *type* F_n if it has a $K(G,1)$ space with finite n-skeleton; and type F if it has one which is a finite complex.

A group has *type* FP_n if there exists a finite sequence of finitely generated projective $\mathbb{Z}G$ modules

$$P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

and is of type FP if there's such sequence which terminates in $0 \to P_n$ for some n.

K can be used to construct a finite projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

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The Bestvina-Brady kernel

Question. Does a group of type FP_n necessarily have type F_n?

Groups of type FP_1 are finitely generated, so have type F_1.

Next step. Are groups of type FP_2 finitely presented?

Define $\phi : G \to \mathbb{Z}$ by $x\phi = 1$, for all $x \in X$. Let $N = \ker(\phi)$.

Theorem (Bestvina & Brady, Invent. Math., '97)

(i) *N is finitely presented if and only if Γ_f is simply connected.*

(ii) *N is of type FP_{n+1} if and only if $H_i(\Gamma_f) = 0$, for all $i \leq n$.*

(iii) *N is of type FP if and only if $H_i(\Gamma_f) = 0$, for all n.*

It's easy to find Γ such that Γ_f satisfies (iii) but is not simply connected; so examples of groups which are not finitely presented but have type FP.

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- Confluent rewriting systems (Hermiller & Meier '95, Bokut & Shiao, '01)
- *Divisibility theory* for G (with many features in common with integers) giving complexity of many main algorithms for the class (Esyp, Kazachkov, Remeslennikov, '05)
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For $Y \subseteq X$ let $Y^\perp = \{z \in X \mid [z, y] = 1, \forall y \in Y\}$.

$L =$ sets of form $(Y^\perp)^\perp$ form a lattice of subsets of X.

Theorem (D, Kazachkov, Remeslennikov)

The maximal length of a chain of centralisers in G is the same as the height of L.

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Automorphisms

- Two graph groups $G(\Gamma_1)$ and $G(\Gamma_2)$ are isomorphic if and only if $\Gamma_1 \cong \Gamma_2$.
- Generators for $\mathrm{Aut}(G)$ are known (Laurence, J.London Math. Soc. '95 , using results of Servatius '89).
- Decomposing $\mathrm{Aut}(G)$ into two parts, one which embeds in $\mathrm{Aut}(G^{\mathrm{ab}})$ and the other for which peak reduction works, Day obtains a presentation for $\mathrm{Aut}(G)$ (arxiv '08).
- Also using peak reduction, Charney, Vogtmann and Crisp have shown that $\mathrm{Out}(G)$ is virtually torsion free, and has finite virtual cohomological dimension. For Γ with no triangles, a finite dimensional, contractible space $O(G)$ on which $\mathrm{Out}(G)$ acts properly is also found. (Geom Topol. '07)
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- Decomposing $\mathrm{Aut}(G)$ into two parts, one which embeds in $\mathrm{Aut}(G^{\mathrm{ab}})$ and the other for which peak reduction works, Day obtains a presentation for $\mathrm{Aut}(G)$ (arxiv '08).
- Also using peak reduction, Charney, Vogtmann and Crisp have shown that $\mathrm{Out}(G)$ is virtually torsion free, and has finite virtual cohomological dimension. For Γ with no triangles, a finite dimensional, contractible space $O(G)$ on which $\mathrm{Out}(G)$ acts properly is also found. (Geom Topol. '07)
- Gutierrez, Piggot and Ruane describe a semidirect product decomposition $(\mathrm{Inn}(G) \rtimes A_1) \rtimes A_2$ of $\mathrm{Aut}(G)$.

Automorphisms cont.

- D, Kazachkov and Remeslennikov have described the structure of certain subgroups of $\mathrm{Aut}(G)$ and show that the stabiliser of the lattice L is an arithmetic subgroup of $\mathrm{Aut}(G)$ (Geom. Dedic. '09).
- Let $\alpha : \mathrm{Aut}(G) \to \mathrm{Aut}(G^{\mathrm{ab}})$ be the canonical map. Noskov shows that $\mathrm{Im}(\alpha)$ is an arithmetic subgroup of $\mathrm{GL}_n(\mathbb{R})$,and also that for some Γ $\mathrm{Aut}(G)$ does not have property T.

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Automorphisms cont.

- D, Kazachkov and Remeslennikov have described the structure of certain subgroups of Aut(G) and show that the stabiliser of the lattice L is an arithmetic subgroup of Aut(G) (Geom. Dedic. '09).
- Let α : Aut(G) $\to$ Aut(G^{ab}) be the canonical map. Noskov shows that Im(α) is an arithmetic subgroup of $\mathrm{GL}_n(\mathbb{R})$,and also that for some Γ Aut(G) does not have property T.