Universal Algebraic Geometry
Part 2

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based on joint results with
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What Is Universal Algebraic Geometry?

Universal algebraic geometry =

= transfer of general notions and ideas from concrete algebraic geometries to the case of arbitrary algebraic structure +

+ formulation and proof of general results not using technique and properties, specific for concrete algebraic structures +

+ development of theory along with decision of new problems arising in this area.

Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{C})$, *i.e.*, $\mathcal{C} \in$ **Ucl**$(\mathcal{A})$;
2. $\mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.

Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathcal{C} \in \textbf{Qvar}(\mathcal{A})$, *i.e.,* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in \textbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.

Definition

An algebra $\mathcal{A}$ is equationally Noetherian, if for any positive integer n and any system of equations $S(x_1, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}(S) = \mathrm{V}(S_0)$.

Language

A first-order language (or signature) is a set

$\mathcal{L} = \{$set of constant symbols $c\} \cup \{$set of functional symbols $F\}$.

Note: We consider only languages with no predicates.

Every functional symbol F is given with their arity n_F .

Terms and Atomic Formulas

Terms in a language $\mathcal{L}$ in variables X are formal expressions defined recursively as follows:

- variables $x_1, x_2, \ldots, x_n$ and constants from $\mathcal{L}$ are terms;
- if $t_1, \ldots, t_n$ are terms and $F(x_1, \ldots, x_n) \in \mathcal{L}$ is function then $F(t_1, \ldots, t_n)$ is a term.

Atomic formulas are formulas of the form $(t = s)$, where t, s are terms.

Formulas in $\mathcal{L}$ in variables X are defined recursively as follows:

- atomic formulas are formulas;
- if Φ and Ψ are formulas then $\neg\Phi, (\Phi \vee \Psi), (\Phi \wedge \Psi), (\Phi \to \Psi)$ are formulas;
- if Φ is a formula and x is a variable then $\forall x\ \Phi$ and $\exists x\ \Phi$ are formulas.

One of the principle results in mathematical logic states that any formula Φ is equivalent to a formula Ψ in the following prenex form:

$$Q_1 x_1 \dots Q_n x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} \Psi_{ij} \right),$$

where $Q_i \in \{\forall, \exists\}$ and Ψ_{ij} is an atomic formula or its negation.

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where $Q_i \in \{\forall, \exists\}$ and Ψ_{ij} is an atomic formula or its negation.

Recall that a universal formula in $\mathcal{L}$ is a formula with no free variables of the type

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right),$$

and a quasi-identity is a universal formula of the type

$$\forall x_1 \dots \forall x_n \left((\bigwedge_{i=1}^{m} t_i(\bar{x}) = s_i(\bar{x})) \quad \to \quad (t(\bar{x}) = s(\bar{x})) \right),$$

where $t, s, t_i, s_i, w_{ij}, v_{ij}$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \dots, x_n)$.

Recall that a universal formula in $\mathcal{L}$ is a formula with no free variables of the type

$$\forall x_1 \ldots \forall x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right),$$

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where $t, s, t_i, s_i, w_{ij}, v_{ij}$ are terms in $\mathcal{L}$ in variables $\bar{x} = (x_1, \ldots, x_n)$.

Examples

Definition

Domain is a associative commutative ring with no zero-devisors and $1 \neq 0$.

Let $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$ be the language of rings.

$$\forall\, x, y \quad x \cdot y = y \cdot x,$$

$$\forall\, x, y \quad x \cdot y = 0 \quad \to \quad x = 0 \vee y = 0,$$

$$1 \neq 0.$$

Fact

These universal formulas axiomatize the universal class of domains in the variety of all associative rings with 1.

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Definition

Group G is a CSA-group if every maximal abelian subgroup H in G is malnormal. Subgroup H in G is malnormal if $H^g \cap H = 1$ for all $g \in G \setminus H$.

Let $\mathcal{L}_{\mathrm{g}} = \{\cdot, ^{-1}, 1\}$ be the language of groups.

$$\text{CT:} \quad \forall\, x, y, z \quad x \neq 1 \wedge [x, y] = [x, z] = 1 \quad \rightarrow \quad [y, z] = 1,$$

$$\forall\, x, y \quad [x^y, x] = 1 \quad \rightarrow \quad [x, y] = 1.$$

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Fact

These universal formulas axiomatize the universal class of CSA-groups in the variety of all groups.

Let k be a field and $\mathcal{L}_{\text{Lie}} = \{+, [,], 0, \alpha\cdot, \alpha \in k\}$ the language of Lie algebras over field k.

Fact

Quasi-identities

$$\forall\, x, y, z, w \quad [[x, y], [z, w]] = 0,$$

$$\forall\, x, y \quad [[x, y], y] = 0 \wedge [[y, x], x] = 0 \quad \to \quad [x, y] = 0,$$

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axiomatize the quasi-variety of metabelian Lie algebras with abelian the Fitting's radicals in the variety of all Lie algebras over field k.

Let $\mathcal{A}$ be an algebra in a language $\mathcal{L}$.

We denote by $\mathrm{Th}_{\forall}(\mathcal{A})$ the set of all universal formulas in $\mathcal{L}$ which hold on algebra $\mathcal{A}$. Similarly, $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ is the set of all quasi-identities in $\mathcal{L}$ which hold on $\mathcal{A}$.

Definition

The set $\mathrm{Th}_{\forall}(\mathcal{A})$ is named universal theory of the algebra $\mathcal{A}$. The set $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ is named quasi-equational theory of the algebra $\mathcal{A}$.

Algebras $\mathcal{A}$ and $\mathcal{B}$ in a language $\mathcal{L}$ are termed universally equivalent, if $\mathrm{Th}_{\forall}(\mathcal{A}) = \mathrm{Th}_{\forall}(\mathcal{B})$. In this case we write $\mathcal{A} \equiv_{\forall} \mathcal{B}$.

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Criterion

Algebraic structures $\mathcal{A}$ and $\mathcal{B}$ in a language $\mathcal{L}$ are universally equivalent iff they have the same local submodels.

- A. I. Malcev, *Algebraic structures*, Nauka, Moscow, 1970.

Definition

The universal closure of $\mathcal{A}$ is the class of all algebras in $\mathcal{L}$ which satisfy all formulas from $\mathrm{Th}_\forall(\mathcal{A})$.

We denote it by **Ucl**$(\mathcal{A})$.

Remark

For two algebras $\mathcal{A}$ and $\mathcal{B}$ in a language $\mathcal{L}$ holds:

$$\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{B}) \iff \mathcal{B} \in \mathbf{Ucl}(\mathcal{A}).$$

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Quasi-variety generated by $\mathcal{A}$ is the class of all algebras in $\mathcal{L}$ which satisfy all formulas from $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$.

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Logical Foundation

There are three perspectives for investigation in algebraic geometry over algebra $\mathcal{A}$:

1. algebraic,
2. geometrical,
3. logic.

Logical approach is connected with examination of universal class **Ucl**$(\mathcal{A})$ and quasi-variety **Qvar**$(\mathcal{A})$.

Theorem (Gorbunov, Maltsev)

1. $\mathbf{Ucl}(\mathcal{A}) = \mathbf{L}(\mathcal{A})$;
2. $\mathbf{Ucl}(\mathcal{A}) = \mathbf{SP_u}(\mathcal{A})$;
3. $\mathbf{Qvar}(\mathcal{A}) = \mathbf{SP_f}(\mathcal{A})$;
4. $\mathbf{Qvar}(\mathcal{A}) = \mathbf{SPP_u}(\mathcal{A}) = \mathbf{SP_uP}(\mathcal{A})$;
5. $\mathbf{Qvar}(\mathcal{A}) = \mathbf{SP_uP}_\omega(\mathcal{A})$;
6. $\mathbf{Qvar}(\mathcal{A}) = \underrightarrow{\mathbf{L}}\mathbf{SP}(\mathcal{A})$;
7. $\mathbf{Qvar}(\mathcal{A}) = \mathbf{S}\underrightarrow{\mathbf{L}}_s\mathbf{P}(\mathcal{A}) = \underrightarrow{\mathbf{L}}_s\mathbf{SP}(\mathcal{A}) = \underrightarrow{\mathbf{L}}_s\mathbf{P_s}(\mathcal{A}) = \underrightarrow{\mathbf{L}}\mathbf{SP}(\mathcal{A})$.

- V. A. Gorbunov, *Algebraic theory of quasivarieties*, Nauchnaya Kniga, Novosibirsk, 1999; English transl., Plenum, 1998.
- A. I. Malcev, *Algebraic structures*, Nauka, Moscow, 1970.

Lemma

Let $\mathcal{A}$ be an equationally Noetherian algebra. Then the following algebras are equationally Noetherian too:

- *every algebra from* $\mathbf{Qvar}(\mathcal{A})$*;*
- *every algebra from* $\mathbf{Ucl}(\mathcal{A})$*.*

Let $\mathcal{A}$ and $\mathcal{B}$ are algebras in a language $\mathcal{L}$.

Definition

We say that algebra $\mathcal{B}$ is separated by algebra $\mathcal{A}$ if for any distinct elements $b_1, b_2 \in B$ there exists a homomorphism $h : \mathcal{B} \to \mathcal{A}$ such that $h(b_1) \neq h(b_2)$.

Definition

We say that an algebra $\mathcal{B}$ is discriminated by algebra $\mathcal{A}$ if for any finite set W of elements from B there exists a homomorphism $h : \mathcal{B} \to \mathcal{A}$ whose restriction onto W is injective.

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Let $F_2 = \langle x, y \rangle$ be the free Lie algebra over a field k of rank 2. Then:

1. $F_2 \times F_2$ is separated by F_2,
2. $F_2 \times F_2$ is not discriminated by F_2.

Facts

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras in a language $\mathcal{L}$. Then:

- if $\mathcal{B}$ is separated by $\mathcal{A}$ then $\mathcal{B} \in \mathbf{Qvar}(\mathcal{A})$;
- if $\mathcal{B}$ is disciminaded by $\mathcal{A}$ then $\mathcal{B} \in \mathbf{Ucl}(\mathcal{A})$.

Let F_r be the free non-abelian group of rank r. Then:

1. $F_2 \subseteq F_r$,
2. $F_2 = \langle x, y \rangle$ contains free group F_∞ of countable rank:

$$F_\infty = \langle\, x^{-1}yx,\ \ x^{-2}yx^2,\ \ x^{-3}yx^3,\ \ldots \rangle,$$

3. thus, F_r is discriminated by F_m for all $r, m \geqslant 2$,
4. thus, $F_r \equiv_\forall F_m$ for all $r, m \geqslant 2$.

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Let F_r be the free non-abelian Lie algebra over a field k of rank r. Then:

1. $F_2 \subseteq F_r$,
2. $F_2 = \langle x, y \rangle$ contains free Lie algebra F_∞ of countable rank:

$$F_\infty = \langle\ [y, x],\ \ [[y, x], x],\ \ [[[y, x], x], x],\ \ldots \rangle,$$

3. thus, F_r is discriminated by F_m for all $r, m \geqslant 2$,
4. thus, $F_r \equiv_\forall F_m$ for all $r, m \geqslant 2$.

Examples

Let F_r be the free non-abelian associative algebra over a field k of rank r. Then:

1. $F_2 \subseteq F_r$,
2. $F_2 = \langle x, y \rangle$ contains free associative algebra F_∞ of countable rank:

$$F_\infty = \langle\, yx, \;\; yxx, \;\; yxxx, \; \ldots \rangle,$$

3. thus, F_r is discriminated by F_m for all $r, m \geqslant 2$,
4. thus, $F_r \equiv_\forall F_m$ for all $r, m \geqslant 2$.

Let F_r be the free non-abelian metabelian Lie algebra over a field k of rank r.

Fact

1. If field k is infinite then F_r is discriminated by F_m and $F_r \equiv_\forall F_m$ for all $r, m \geqslant 2$.
2. It field k is finite then F_r is not discriminated by F_m and $F_r \not\equiv_\forall F_m$ for all $r > m \geqslant 2$.

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras in a language $\mathcal{L}$.

Definition

We say that algebras $\mathcal{A}$ and $\mathcal{B}$ are geometrically equivalent if for any natural number n and any system of equations $S(x_1, \ldots, x_n)$ the equality

$$\mathrm{Rad}_{\mathcal{A}}(S) = \mathrm{Rad}_{\mathcal{B}}(S)$$

holds.

B.I. Plotkin asked: When to algebras $\mathcal{A}$ and $\mathcal{B}$ are geometrically equivalent?

- B. I. Plotkin, *Algebras with the same (algebraic) geometry*, Tr. Mat. Inst. Steklova, **242**, 2003, pp. 176–207.

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Lemma

The following conditions are equivalent:

1. *$\mathcal{A}$ and $\mathcal{B}$ are geometrically equivalent;*
2. *the class of coordinate algebras of algebraic sets over $\mathcal{A}$ coincides with the class of coordinate algebras of algebraic sets over $\mathcal{B}$;*
3. $\mathbf{Res}(\mathcal{A})_\omega = \mathbf{Res}(\mathcal{B})_\omega$.

$\mathbf{Res}(\mathcal{A})_\omega$ = class of finitely generated algebras which separated by algebra $\mathcal{A}$.

Corollary

Let $\mathcal{A}$ and $\mathcal{B}$ be equationally Noetherian algebras in a language $\mathcal{L}$. Then the following conditions are equivalent:

1. *$\mathcal{A}$ and $\mathcal{B}$ are geometrically equivalent;*
2. $\mathbf{Qvar}(\mathcal{A}) = \mathbf{Qvar}(\mathcal{B})$.

Geometrical Equivalence

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Lemma

The following conditions are equivalent:

1. $\mathcal{A}$ *and* $\mathcal{B}$ *are geometrically equivalent;*
2. *the class of coordinate algebras of algebraic sets over* $\mathcal{A}$ *coincides with the class of coordinate algebras of algebraic sets over* $\mathcal{B}$*;*
3. $\mathbf{Res}(\mathcal{A})_\omega = \mathbf{Res}(\mathcal{B})_\omega$.

$\mathbf{Res}(\mathcal{A})_\omega$ = class of finitely generated algebras which separated by algebra $\mathcal{A}$.

Corollary

Let $\mathcal{A}$ *and* $\mathcal{B}$ *be equationally Noetherian algebras in a language* $\mathcal{L}$*. Then the following conditions are equivalent:*

1. $\mathcal{A}$ *and* $\mathcal{B}$ *are geometrically equivalent;*
2. $\mathbf{Qvar}(\mathcal{A}) = \mathbf{Qvar}(\mathcal{B})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras in a language $\mathcal{L}$.

Definition

Algebras $\mathcal{A}$ and $\mathcal{B}$ are universally geometrically equivalent if

1. the class of coordinate algebras over $\mathcal{A}$ coincides with the class of coordinate algebras over $\mathcal{B}$;
2. the class of irreducible coordinate algebras over $\mathcal{A}$ coincides with the class of irreducible coordinate algebras over $\mathcal{B}$.

Lemma

The following conditions are equivalent:

1. $\mathcal{A}$ *and* $\mathcal{B}$ *are universal geometrically equivalent;*
2. $\mathbf{Dis}(\mathcal{A})_\omega = \mathbf{Dis}(\mathcal{B})_\omega$.

$\mathbf{Dis}(\mathcal{A})_\omega$ = class of finitely generated algebras which discriminated by algebra $\mathcal{A}$.

Corollary

Let $\mathcal{A}$ *and* $\mathcal{B}$ *be equationally Noetherian algebras in a language* $\mathcal{L}$. *Then the following conditions are equivalent:*

1. $\mathcal{A}$ *and* $\mathcal{B}$ *are geometrically equivalent;*
2. $\mathcal{A} \equiv_\forall \mathcal{B}$.

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Fact

All abelian torsion-free groups are universal equivalent to each other. So, they have the same algebraic geometry.

Let $\mathcal{A}$ be an algebra in a language $\mathcal{L}$.

Definition

Prevariety **Pvar**$(\mathcal{A})$ generated by algebra $\mathcal{A}$ is the least class of algebras in $\mathcal{L}$, such that:

- it contains algebra $\mathcal{A}$;
- it closed under direct products;
- it closed under subalgebras.

Negations of universal formulas are existential formulas:

$$\exists x_1 \dots \exists x_n \left(\bigwedge_{i=1}^{m} \bigvee_{j=1}^{s_i} w_{ij}(\bar{x}) \overset{=}{\neq} v_{ij}(\bar{x}) \right).$$

We denote by $\mathrm{Th}_{\exists}(\mathcal{A})$ the existential theory of algebra $\mathcal{A}$ =
= the set of all existential formulas in $\mathcal{L}$ which hold on $\mathcal{A}$.

Fact

For two algebras $\mathcal{A}$ and $\mathcal{B}$ in a language $\mathcal{L}$ hold:

- $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{B}) \quad \Longrightarrow \quad \mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{B})$;
- $\mathcal{A} \equiv_{\forall} \mathcal{B} \quad \Longleftrightarrow \quad \mathcal{A} \equiv_{\exists} \mathcal{B}$.

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Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{C})$, *i.e.*, $\mathcal{C} \in$ **Ucl**$(\mathcal{A})$;
2. $\mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.

Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, *i.e.*, $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.