Universal Algebraic Geometry
Part 1

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based on joint results with
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Algebraic geometry over algebraic structures (= Universal Algebraic Geometry) is a new area of researching in algebra today, but it has a deep genealogy. The equations played important role in mathematics at all times.

Statement (Remeslennikov)

No Equations $\Longrightarrow$ No Mathematics.

Algebraic geometry as mathematical discipline starts from 19th century. Classical algebraic geometry studies equations and their solutions over fields (the real number field, the complex number field, algebraic closed fields, finite fields, and so on).

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New needs and new problems in contemporary mathematics have motivated to transfer classical algebraic geometry theory to another algebraic structures (groups, rings, algebras over a ring, semigroups, and so on). This generalization was started in work:

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- B. I. Plotkin, *Some concepts of algebraic geometry in uneversal algebra*, Algebra i Analiz, **9 (4)**, 1997, pp. 224–248.
- B. I. Plotkin, *Algebras with the same (algebraic) geometry*, Tr. Mat. Inst. Steklova, **242**, 2003, pp. 176–207.

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At the present time the largest developments we can see in algebraic geometry over groups where the most striking result is the solution of the main problem of algebraic geometry on classification of algebraic sets and coordinate groups in the case of the free group. The classification of coordinate groups is given in the language of free constructions. This was obtained due to works of many specialists in the group theory.

- O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free group I: Irreducibility of quadratic equations and Nullstellensatz*, J. Algebra, **200 (2)**, 1998, pp. 472–516.
- O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over free group II: Systems in trangular quasi-quadratic form and description of residually free groups*, J. Algebra, **200(2)**, 1998, pp. 517–570.
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Outstanding Results

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What Is Universal Algebraic Geometry?

Universal algebraic geometry =

= transfer of general notions and ideas from concrete algebraic geometries to the case of arbitrary algebraic structure +

+ formulation and proof of general results without help of technique and properties, specific for concrete algebraic structures +

+ development of theory along with decision of new problems arising in this area.

Language

A first-order language (or signature) is a set

$\mathcal{L} = \{$set of constant symbols $c\} \cup \{$set of functional symbols $F\}$.

Note: We consider only languages with no predicates.

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Algebras

An algebraic structure $\mathcal{A}$ in a language $\mathcal{L}$ is given by the following data:

- a non-empty set A called the universe of $\mathcal{A}$;
- interpretation of symbols from $\mathcal{L}$ on the universe A:
  - a function $F : A^{n_F} \to A$ of arity n_F for each function $F \in \mathcal{L}$;
  - an element $c \in A$ for each constant $c \in \mathcal{L}$.

We use notation $A, B, C, \dots$ to refer to the universes of the structures $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$.

Algebraic structures in a language with no predicates are termed algebras.

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Let $X = \{x_1, \dots, x_n\}$ be a finite set of variables.

Definition

Equation in the language $\mathcal{L}$ in variables X is an atomic formula

$$(t = s),$$

where t, s are terms.

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Terms in a language $\mathcal{L}$ in variables X are formal expressions defined recursively as follows:

- variables $x_1, x_2, \dots, x_n$ and constants from $\mathcal{L}$ are terms;
- if $t_1, \dots, t_n$ are terms and $F(x_1, \dots, x_n) \in \mathcal{L}$ is function then $F(t_1, \dots, t_n)$ is a term.

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Terms and Atomic Formulas

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Atomic formulas are formulas of the form $(t = s)$, where t, s are terms.

Let $\mathcal{A}$ be an algebra in a language $\mathcal{L}$ and S a system of equations in $\mathcal{L}$.

Definition

The solution of a system of equations S over $\mathcal{A}$,

$$\mathrm{V}(S) = \{ \, (a_1, \ldots, a_n) \in A^n \mid \\ t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n) \quad \forall \, (t = s) \in S \, \},$$

is termed the algebraic set over $\mathcal{A}$.

Radicals

Definition

The set of atomic formulas

$$\mathrm{Rad}(S) = \{\ (t = s) \mid t(a_1, \ldots, a_n) = s(a_1, \ldots, a_n) \quad \forall\ (a_1, \ldots, a_n) \in \mathrm{V}(S)\ \}$$

is termed the radical of the algebraic set $\mathrm{V}(S)$.

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is termed the radical of the algebraic set $\mathrm{V}(S)$.

$\mathrm{Rad}(S)$ is the maximal system of equations which equivalent to S.

Let $Y = \mathrm{V}(S)$. Radical $\mathrm{Rad}(S)$ defines some congruence on the absolutely free algebra (the algebra of terms) in the language $\mathcal{L}$ in variables X:

$$t \sim s \quad \Longleftrightarrow \quad (t = s) \in \mathrm{Rad}(S), \quad t, s \text{ are terms.}$$

Definition

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	Algebra	Signature	Typical form of equations S	Radical or algebraic set	Coordinate algebra
1	Free group $F=\langle a_1,a_2,\dots,a_n\rangle$	$\{\cdot,{}^{-1},1,a_1,a_2\}$	$x^2y^2z^2=1$	$V(S)=\{(f_1,f_2,f_3)\mid [f_1,f_2]=[f_2,f_3]=[f_3,f_1]=1$ $f_1^2f_2^2f_3^2=1\}$	$\Gamma(S)\simeq F*\mathbb{Z}^3$
			$[x,y]=[a_1,a_2]$	$V(S)$ is a very complex set	$\Gamma(S)=F*_{[x,y]=[a_1,a_2]}\langle x,y\rangle$ is a free amalgamated product
2	Additive monoid of natural numbers $\mathbb{N}$	$\{+,0,1\}$	$3x+4y+7z=7$	$V(S)=\{(0,0,1),(1,1,0)\}$	$\Gamma(S)=\mathbb{N}\oplus\langle x\rangle\oplus\langle z\rangle / \langle x+z=1\rangle$
3	Free Lie algebra over a field k $L=\langle a,b\rangle$	$\{+,[\,,],a,b\}$	$[x,a]+[y,b]=0$	*Remeslennikov, Stöhr:* $V(S)$ is an infinite dimensional linear space over k	$\Gamma(S)=L*\langle x,y\rangle / \mathrm{id}\langle [x,a]+[x,b]\rangle$
			$[[x,a],[b,a]]=0$	*Remeslennikov, Daniyarova:* $V(S)=\mathrm{lin}_k\{a,b\}$	$\Gamma(S)\simeq\langle L,t_1a+t_2b\rangle$, where $t_1,t_2\in\prod_{x\in L}k^{(x)}$ are algebraic independent over k
4	Linear space over a field k V	$\{+,\alpha\cdot,\alpha\in k, v,v\in V\}$	$\alpha x+\beta y+\dots+v=0$ $\alpha,\beta\in k,\ v\in V$	$x_1,\dots,x_r$ are free variables, $y_1,\dots,y_m$ are dependent variables, $V(S)=\{(\bar{x},\bar{y})\mid\bar{y}=A\bar{x}+\bar{b}\}$	$\Gamma(S)\simeq V\oplus\mathrm{lin}_k\langle x_1,\dots,x_r\rangle$

There are three segments in concrete algebraic geometries:

- Coefficient-free algebraic geometry;
- Diophantine algebraic geometry;
- Algebraic geometry with coefficients in some algebra $\mathcal{A}$ and solutions in some extension $\mathcal{A} < \mathcal{B}$ (usually, in saturated model).

Universal Approach

From the point of view of Universal Algebraic Geometry these three segments may be examined by means of one universal technique. It is just sufficient to make suitable choice of signature $\mathcal{L}$.

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Universal Approach

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Example

Let us consider the language of groups $\mathcal{L}_{\mathrm{g}} = \{\cdot, ^{-1}, e\}$, a group G and some extension $G < H$. Then:

- Coefficient-free algebraic geometry over G is algebraic geometry over G in the language $\mathcal{L}_{\mathrm{g}}$;
- Diophantine algebraic geometry over G is algebraic geometry over G in the extended language

$$\mathcal{L}_{\mathrm{g},G} = \mathcal{L}_{\mathrm{g}} \cup \{c_g, g \in G\};$$

- Algebraic geometry over H with coefficients in G is algebraic geometry over H in the extended language $\mathcal{L}_{\mathrm{g},G}$.

Major Problem

One of the major problems of algebraic geometry over given an algebra $\mathcal{A}$ in a language $\mathcal{L}$ consists in classifying algebraic sets over $\mathcal{A}$ with accuracy up to isomorphism.

One can classify algebraic sets by means of three languages, which are equivalent to each other:

1. in geometric language, by describing algebraic sets directly;
2. in the language of radical ideals;
3. and in algebraic language, by classifying coordinate algebras of algebraic sets.

Fact

Every algebraic set may be restored in unique manner from its radical and it may be restored from its coordinate structure just up to isomorphism.

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Fact

Every algebraic set may be restored in unique manner from its radical and it may be restored from its coordinate structure just up to isomorphism.

Let $\mathcal{A}$ be an algebra in a language $\mathcal{L}$.

Theorem

The category of algebraic sets over algebra $\mathcal{A}$ and the category of coordinate algebras of algebraic sets over $\mathcal{A}$ are dually equivalent.

- E. Daniyarova, A. Miasnikov, V. Remeslennikov, *Unification theorems in algebraic geometry*, Algebra and Discrete Mathematics, **1** (2008), and on arxiv.org.

First Universal Result

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Theorem

Let $\mathcal{A}$ be an equationally Noetherian algebra in a language $\mathcal{L}$ (with no predicates). Then for a finitely generated algebra $\mathcal{C}$ of $\mathcal{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_\forall(\mathcal{A}) \subseteq \mathrm{Th}_\forall(\mathcal{C})$, *i.e.,* $\mathcal{C} \in \mathbf{Ucl}(\mathcal{A})$;
2. $\mathrm{Th}_\exists(\mathcal{A}) \supseteq \mathrm{Th}_\exists(\mathcal{C})$;
3. $\mathcal{C}$ *embeds into an ultrapower of* $\mathcal{A}$;
4. $\mathcal{C}$ *is discriminated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a limit algebra over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty irreducible algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.

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1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, *i.e.,* $\mathrm{Th}_{qi}(\mathcal{A}) \subseteq \mathrm{Th}_{qi}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ *embeds into a direct power of* $\mathcal{A}$;
4. $\mathcal{C}$ *is separated by* $\mathcal{A}$;
5. $\mathcal{C}$ *is a subdirect product of finitely many limit algebras over* $\mathcal{A}$;
6. $\mathcal{C}$ *is an algebra defined by a complete atomic type in the theory* $\mathrm{Th}_{qi}(\mathcal{A})$ *in* $\mathcal{L}$;
7. $\mathcal{C}$ *is the coordinate algebra of a non-empty algebraic set over* $\mathcal{A}$ *defined by a system of equations in the language* $\mathcal{L}$.

Definition

An algebra $\mathcal{A}$ is equationally Noetherian, if for any positive integer n and any system of equations $S(x_1, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $\mathrm{V}(S) = \mathrm{V}(S_0)$.

Remark

In above definition the information on some language $\mathcal{L}$ is hidden. When saying that some group G is equationally Noetherian it is necessary to point out the implicit language.

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	Equationally Noetherian: Yes or No?	
1	Noetherian commutative ring	Yes
2	Linear group over Noetherian ring (free group, polycyclic group, finitely generated metabelian group)	Yes
3	Torsion-free hyperbolic group	Yes
4	Free solvable group	Yes
5	Finitely generated metabelian (or nilpotent) Lie algebra	Yes
6	Infinitely generated nilpotent group	No
7	Wreath product $A \wr B$ of a non-abelian group A and an infinite group B	No
8	The Grigorchuk group Γ	No
9	The min-max structures $\langle \mathbb{R}; \max, \min, +, -, \cdot, 0, 1 \rangle$ and $\langle \mathbb{N}; \max, \min, +, 0, 1 \rangle$	No
10	Free Lie algebra	?
11	Free anti-commutative algebra	?
12	Free associative algebra	?
13	Free products of equationally Noetherian groups	?

Lemma

Let $\mathcal{A}$ be an equationally Noetherian algebra. Then the following algebras are equationally Noetherian too:

1. *every subalgebra of $\mathcal{A}$;*
2. *every filterpower (ultrapower, direct power) of $\mathcal{A}$;*
3. *coordinate algebra $\Gamma(Y)$ of an algebraic set Y over $\mathcal{A}$;*

Corollary

- *every algebra from* $\mathbf{Qvar}(\mathcal{A})$;
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There are three perspectives for investigation in algebraic geometry over algebra $\mathcal{A}$: algebraic, geometrical and logic. Geometrical approach is connected with examination of affine space A^n as topological space.

We define Zariski topology on A^n, where algebraic sets over $\mathcal{A}$ form a subbase of closed sets, i.e., closed sets in this topology are obtained from the algebraic sets by finite unions and (arbitrary) intersections.

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Lemma

For an algebra $\mathcal{A}$ the following conditions are equivalent:

1. *$\mathcal{A}$ is equationally Noetherian;*
2. *for any natural number n the Zariski topology on A^n is Noetherian (satisfies the descending chain condition on closed subsets);*
3. *every chain*

$$\Gamma(Y_1) \to \Gamma(Y_2) \to \Gamma(Y_3) \to \dots$$

   of proper epimorphisms of coordinate algebras of algebraic sets $Y_1, Y_2, Y_3, \dots$ over $\mathcal{A}$ is finite.

Theorem

Every algebraic set Y over equationally Noetherian algebra $\mathcal{A}$ can be expressed as a finite union of irreducible algebraic sets (irreducible components):

$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m.$$

Furthermore, this decomposition is unique up to permutation of irreducible components and omission of superfluous ones.

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Classification of irreducible algebraic sets (and/or their coordinate algebras) is the essential problem of algebraic geometry over equationally Noetherian algebra.

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