Lyndon's Completion for Partially Commutative Groups, Part 2

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Outline

1. Construction of the completion;
2. Structure of the completion;
3. Outlook.

Construction: free group case

- Let F be a free group and let $C(v)$ be the centraliser of an element v from F.
- Consider the extension of the centraliser C,

$$H = \langle F, t \mid [c, t] = 1, \text{ for all } c \in C \rangle.$$

- Since F has BP property, H is F-discriminated by F.
- Since H is F-discriminated by F and F has BP property, H has BP property.
- Repeating this procedure, one gets a chain of embeddings:

$$F = H_0 \hookrightarrow H_1 \hookrightarrow \ldots \hookrightarrow H_k \hookrightarrow \ldots$$

- We define $\widetilde{F}$ to be the direct limit of $\{H_i\}$.

Construction: partially commutative groups

- On one hand, we can not expect to extend all centralisers- Ilya's contraexample.
- But on the other hand, we proved that p.c.groups have BP property w.r.t block elements:

$$\forall b_1, \dots, b_k \text{ block elements s.t. } [b_i, b_{i+1}] \neq 1,$$

$$\exists n_0 \text{ s.t. } \forall n_1, \dots n_k \geq n_0, b_1^{n_1} \cdots b_k^{n_k} \neq 1$$

- The idea is simple: extend ONLY centralisers of block elements.

Construction: partially commutative groups

- Let G be a partially commutative group and let $C(b)$ be the centraliser of a BLOCK element b from G.
- Consider the extension of the centraliser C,

$$H = \langle G, t \mid [c, t] = 1, \text{ for all } c \in C \rangle.$$

- Since G has BP property, H is G-discriminated by G.
- But the fact that H is G-discriminated by G and that G has BP property does NOT imply that H has BP property.

Construction: partially commutative groups

- We need a stronger condition- block discrimination:

$$\forall h_1, \ldots, h_k \text{ block elements in } H \text{ and}$$

$$\forall h_{k+1}, \ldots, h_l \in H,$$

  there exists a G-homomorphism $\phi : H \to G$ such that:

  1. ϕ is infective in $h_1, \ldots h_l$ and
  2. $\phi(h_i)$ is a block in G, for all $i = 1, \ldots, k$.

- the proof is too technical...

Construction: partially commutative groups

- First attempt: try to characterize block elements in a more "logical way".
- It did not work :(
- Second attempt: try to avoid the elements whose centralisers can not be extended - introduce basic elements. A c.r. element b is called *basic* if

$$''\forall g_1, g_2 \in G, [g_1, b] \neq 1, [g_2, b] \neq 1 \to [g_1^{b^N}, g_2] \neq 1''.$$

  $(N = N(G))$

- Now it is easy to prove that if H is G-discriminated by G then it is in fact marked-discriminated ("basics go to basics").

Construction: partially commutative groups

- So, is H G-discriminated by G?
- We gave a characterisation of the set of basic elements:
- block elements belong to the set :)
- but also the tuple of elements that provide a counterexample for the BP property!!!!!

Construction: partially commutative groups

- Let us analize the contraexample again...
- BP is a sufficient condition for discrimination, but not necessary
- let us “weaken” it - the discrimination condition
- FINALLY, we proved that p.c. groups have the discrimination condition w.r.t basic elements.
- So, the extension of a centraliser of a basic element is discriminated and the group has the discrimination property.
- We can now recursively construct our group $\widetilde{G}$, which is G-discriminated by G.

Construction: ( )

- Let us repeat in a more abstract way what we did...
- This idea, although naive it is turns out that can be used in other cases.
- Consider a relatively hyperbolic group G and mark the set of hyperbolic elements.
- It follows from Osin's work, that relatively hyperbolic groups have BP property w.r.t hyperbolic elements. Therefore the group constructed from an extension of a centraliser of a hyperbolic element is discriminated.
- Furthermore, it also follows from Osin's work that, in fact, the discriminating family is marked-discriminating.
- We can recursively repeat the construction.

Construction: ( )

- Consider G to be the free product of groups:
- This construction gives a way to describe some fully residually G-groups.
- What is the relation between the completion of G and the completion of the factors? Is the "difference" covered by extension of centralisers of hyperbolic elements?

Structure of $\widetilde{G}$: free groups

- Free groups are CSA and therefore centralisers of elements are abelian.
- The CSA property is invariant w.r.t extensions of centralisers and transfinite induction. Hence $\widetilde{G}$ is also CSA.
- Centralisers of elements from $\widetilde{G}$ are one-generated $\mathbb{Z}[t]$-modules. This implies that they coincide with the tensor completion. In particular, they satisfy the universal property.
- One can define naturally the action of $\mathbb{Z}[t]$ on the centralisers.
- Using properties of the action and the CSA property, one can extend the action to $\widetilde{G}$.
- The universal property of the centralisers extend to $\widetilde{G}$.
- One can conclude that $\widetilde{G}$ is the tensor completion of the group G.

Structure of $\widetilde{G}$: partially commutative groups

We need to understand the abstract properties of G that give rigidity to the structure of the centralisers of elements from H_i.

Structure of $\widetilde{G}$: partially commutative groups

We define

1. block structure:
   Let $\mathfrak{B} \subset G$ such that $1 \notin \mathfrak{B}$. $(G, \mathfrak{B})$ is *a block structure* if $\forall g \in G, g \neq 1$ there exists $\{b_1, \dots, b_k\} \subset \mathfrak{B}$ s.t. $g = b_1 \cdots b_k$. Furthermore,
   (Uniqueness) For all $g \in G$ the block decomposition is unique;
   (Commutativity) $\forall b_1, \dots, \forall b_k \in \mathfrak{B}$ if $[b_i, b_j] = 1$ and $b_i^{-1} \neq b_j$ $i, j = 1, \dots, k$ implies that $C(b_1 \cdots b_k) = \bigcap_{i=1,\dots,k} C(b_i)$.

   Let $(G, \mathfrak{B})$ be a block structure. Then for any $b \in \mathfrak{B}$ the centraliser $C(b) \simeq \langle b \rangle \times A_b$.
2. WCSA: G is WCSA if for all block elements b and for all $g \in G$ s.t. $[g, b] \neq 1$, $b^g \notin C(b)$.

Structure of $\widetilde{G}$: partially commutative groups

- The BS and WCSA properties are invariant w.r.t extensions of centralisers and transfinite induction. Hence $\widetilde{G}$ also has BS and WCSA properties.
- The centraliser of a block element g from $\widetilde{G}$ is a direct products of a one-generated $\mathbb{Z}[t]$-module and $\widetilde{A}_g$. Using induction on the rank of $A_g \cap G$, we conclude that $C(g)$ coincides with the tensor completion and that it satisfies the universal property.
- One can define naturally the action of $\mathbb{Z}[t]$ on the one-generated $\mathbb{Z}[t]$-module direct factor of the centraliser.
- Using properties of the action and the WCSA property, one can extend the action to $\widetilde{G}$.
- The universal property of the centralisers extend to $\widetilde{G}$.
- One can conclude that $\widetilde{G}$ is the tensor completion of the group G.

Statement of the results

Theorem

Let G be a pc group. Then the tensor completion of G by $\mathbb{Z}[t]$ is the complete tensor extension of centralisers of G by the ring $\mathbb{Z}[t]$.

Theorem

Let G be a pc group and let $G^{\mathbb{Z}[t]}$ be its tensor completion. Then $G^{\mathbb{Z}[t]}$

- *is discriminated by G;*
- *is torsion free;*
- *has block structure;*
- *is WCSA;*
- *has G as a subgroup;*
- *has reduced and canonical forms etc.*

Note: the results are not stated in the full generality.

Outlook

- Embedding of all f.g. fully residually G groups into $G^{\mathbb{Z}[t]}$ and their structure.
- Develop infinite words to solve algorithmic problems for (some) f.g. f. r. G groups.
- Generalisations to other groups: graph products, etc.