

Lyndon's Completion for Partially Commutative Groups, Part 2

Montserrat Casals-Ruiz, Ilya Kazachkov

August 21, 2009

Outline

1. Construction of the completion;
2. Structure of the completion;
3. Outlook.

Construction: free group case

- ▶ Let F be a free group and let $C(v)$ be the centraliser of an element v from F .
- ▶ Consider the extension of the centraliser C ,

$$H = \langle F, t \mid [c, t] = 1, \text{ for all } c \in C \rangle.$$

- ▶ Since F has BP property, H is F -discriminated by F .
- ▶ Since H is F -discriminated by F and F has BP property, H has BP property.
- ▶ Repeating this procedure, one gets a chain of embeddings:

$$F = H_0 \hookrightarrow H_1 \hookrightarrow \dots \hookrightarrow H_k \hookrightarrow \dots$$

- ▶ We define \tilde{F} to be the direct limit of $\{H_i\}$.

Construction: partially commutative groups

- ▶ On one hand, we can not expect to extend all centralisers-Ilya's contraexample.
- ▶ But on the other hand, we proved that p.c.groups have BP property w.r.t block elements:

$\forall b_1, \dots, b_k$ block elements s.t. $[b_i, b_{i+1}] \neq 1$,

$\exists n_0$ s.t. $\forall n_1, \dots, n_k \geq n_0, b_1^{n_1} \cdots b_k^{n_k} \neq 1$

- ▶ The idea is simple: extend ONLY centralisers of block elements.

Construction: partially commutative groups

- ▶ Let G be a partially commutative group and let $C(b)$ be the centraliser of a BLOCK element b from G .
- ▶ Consider the extension of the centraliser C ,

$$H = \langle G, t \mid [c, t] = 1, \text{ for all } c \in C \rangle.$$

- ▶ Since G has BP property, H is G -discriminated by G .
- ▶ But the fact that H is G -discriminated by G and that G has BP property does NOT imply that H has BP property.

Construction: partially commutative groups

- ▶ We need a stronger condition- block discrimination:

$\forall h_1, \dots, h_k$ block elements in H and

$\forall h_{k+1}, \dots, h_l \in H,$

there exists a G -homomorphism $\phi : H \rightarrow G$ such that:

1. ϕ is injective in h_1, \dots, h_l and
 2. $\phi(h_i)$ is a block in G , for all $i = 1, \dots, k$.
- ▶ the proof is too technical...

Construction: partially commutative groups

- ▶ First attempt: try to characterize block elements in a more “logical way”.
- ▶ It did not work :(
- ▶ Second attempt: try to avoid the elements whose centralisers can not be extended - introduce basic elements. A c.r. element b is called *basic* if

$$" \forall g_1, g_2 \in G, [g_1, b] \neq 1, [g_2, b] \neq 1 \rightarrow [g_1^{b^N}, g_2] \neq 1 " .$$

$$(N = N(G))$$

- ▶ Now it is easy to prove that if H is G -discriminated by G then it is in fact marked-discriminated (“basics go to basics”).

Construction: partially commutative groups

- ▶ So, is H G -discriminated by G ?
- ▶ We gave a characterisation of the set of basic elements:
- ▶ block elements belong to the set :)
- ▶ but also the tuple of elements that provide a counterexample for the BP property!!!!

Construction: partially commutative groups

- ▶ Let us analyze the contraexample again...
- ▶ BP is a sufficient condition for discrimination, but not necessary
- ▶ let us “weaken” it - the discrimination condition
- ▶ FINALLY, we proved that p.c. groups have the discrimination condition w.r.t basic elements.
- ▶ So, the extension of a centraliser of a basic element is discriminated and the group has the discrimination property.
- ▶ We can now recursively construct our group \tilde{G} , which is G -discriminated by G .

Construction: ()

- ▶ Let us repeat in a more abstract way what we did...
- ▶ This idea, although naive it is turns out that can be used in other cases.
- ▶ Consider a relatively hyperbolic group G and mark the set of hyperbolic elements.
- ▶ It follows from Osin's work, that relatively hyperbolic groups have BP property w.r.t hyperbolic elements. Therefore the group constructed from an extension of a centraliser of a hyperbolic element is discriminated.
- ▶ Furthermore, it also follows from Osin's work that, in fact, the discriminating family is marked-discriminating.
- ▶ We can recursively repeat the construction.

Construction: ()

- ▶ Consider G to be the free product of groups:
- ▶ This construction gives a way to describe some fully residually G -groups.
- ▶ What is the relation between the completion of G and the completion of the factors? Is the “difference” covered by extension of centralisers of hyperbolic elements?

Structure of \tilde{G} : free groups

- ▶ Free groups are CSA and therefore centralisers of elements are abelian.
- ▶ The CSA property is invariant w.r.t extensions of centralisers and transfinite induction. Hence \tilde{G} is also CSA.
- ▶ Centralisers of elements from \tilde{G} are one-generated $\mathbb{Z}[t]$ -modules. This implies that they coincide with the tensor completion. In particular, they satisfy the universal property.
- ▶ One can define naturally the action of $\mathbb{Z}[t]$ on the centralisers.
- ▶ Using properties of the action and the CSA property, one can extend the action to \tilde{G} .
- ▶ The universal property of the centralisers extend to \tilde{G} .
- ▶ One can conclude that \tilde{G} is the tensor completion of the group G .

Structure of \tilde{G} : partially commutative groups

We need to understand the abstract properties of G that give rigidity to the structure of the centralisers of elements from H_j .

Structure of \tilde{G} : partially commutative groups

We define

1. block structure:

Let $\mathfrak{B} \subset G$ such that $1 \notin \mathfrak{B}$. (G, \mathfrak{B}) is a *block structure* if $\forall g \in G, g \neq 1$ there exists $\{b_1, \dots, b_k\} \subset \mathfrak{B}$ s.t. $g = b_1 \cdots b_k$. Furthermore,

(Uniqueness) For all $g \in G$ the block decomposition is unique;

(Commutativity) $\forall b_1, \dots, \forall b_k \in \mathfrak{B}$ if $[b_i, b_j] = 1$ and $b_i^{-1} \neq b_j$ $i, j = 1, \dots, k$ implies that $C(b_1 \cdots b_k) = \bigcap_{i=1, \dots, k} C(b_i)$.

Let (G, \mathfrak{B}) be a block structure. Then for any $b \in \mathfrak{B}$ the centraliser $C(b) \simeq \langle b \rangle \times A_b$.

2. WCSA: G is WCSA if for all block elements b and for all $g \in G$ s.t. $[g, b] \neq 1, b^g \notin C(b)$.

Structure of \tilde{G} : partially commutative groups

- ▶ The BS and WCSA properties are invariant w.r.t extensions of centralisers and transfinite induction. Hence \tilde{G} also has BS and WCSA properties.
- ▶ The centraliser of a block element g from \tilde{G} is a direct products of a one-generated $\mathbb{Z}[t]$ -module and \tilde{A}_g . Using induction on the rank of $A_g \cap G$, we conclude that $C(g)$ coincides with the tensor completion and that it satisfies the universal property.
- ▶ One can define naturally the action of $\mathbb{Z}[t]$ on the one-generated $\mathbb{Z}[t]$ -module direct factor of the centraliser.
- ▶ Using properties of the action and the WCSA property, one can extend the action to \tilde{G} .
- ▶ The universal property of the centralisers extend to \tilde{G} .
- ▶ One can conclude that \tilde{G} is the tensor completion of the group G .

Statement of the results

Theorem

Let G be a pc group. Then the tensor completion of G by $\mathbb{Z}[t]$ is the complete tensor extension of centralisers of G by the ring $\mathbb{Z}[t]$.

Theorem

Let G be a pc group and let $G^{\mathbb{Z}[t]}$ be its tensor completion. Then $G^{\mathbb{Z}[t]}$

- ▶ is discriminated by G ;
- ▶ is torsion free;
- ▶ has block structure;
- ▶ is WCSA;
- ▶ has G as a subgroup;
- ▶ has reduced and canonical forms etc.

Note: the results are not stated in the full generality.

Outlook

- ▶ Embedding of all f.g. fully residually G groups into $G^{\mathbb{Z}[t]}$ and their structure.
- ▶ Develop infinite words to solve algorithmic problems for (some) f.g. f. r. G groups.
- ▶ Generalisations to other groups: graph products, etc.