Conjugacy separability

O. Bogopolski

Omsk, 21.08.09

On uniform conjugators in torsion-free hyperbolic groups

On conjugacy separability for subgroups of virtually free groups

On uniform conjugators in torsion-free hyperbolic groups
(O. Bogopolski, E. Ventura)

Definition. Let G be a group and A be a subset of G. An endomorphism φ of G is called *pointwise inner on* A if $\varphi(a)$ is conjugate to a for every $a \in A$.

On uniform conjugators in torsion-free hyperbolic groups

(O. Bogopolski, E. Ventura)

Definition. Let G be a group and A be a subset of G.
An endomorphism φ of G is called *pointwise inner on A* if $\varphi(a)$ is conjugate to a for every $a \in A$.

We call φ *pointwise inner* if it is pointwise inner on G.

On uniform conjugators in torsion-free hyperbolic groups
(O. Bogopolski, E. Ventura)

Definition. Let G be a group and A be a subset of G.
An endomorphism φ of G is called *pointwise inner on A* if $\varphi(a)$ is conjugate to a for every $a \in A$.

We call φ *pointwise inner* if it is pointwise inner on G.

$\mathrm{Aut}_{\mathrm{pi}}(G)$ – the group of all pointwise inner automorphisms of G.
Clearly,

$$\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}_{\mathrm{pi}}(G) \trianglelefteq \mathrm{Aut}(G).$$

Groups, which admit pointwise inner but non-inner automorphisms

1. some finite groups (D. Robinson),
2. any free nilpotent group of class $c \geqslant 3$ (G. Endimioni),
3. some nilpotent Lie groups (C.S. Gordon, E.N. Wilson; they used that to construct isospectral but not isometric Riemannian manifolds)
4. direct products of such groups with arbitrary groups.

Groups, whose pointwise inner automorphisms are inner

1. free groups (A. Lubotzky),
2. non-trivial free products (M.V. Neshadim),
3. fundamental groups of closed surfaces of negative Euler characteristic
   (O. Bogopolski, E. Kudrjavtseva, H.Zieschang)
4. torsion-free hyperbolic groups
   (O. Bogopolski, E. Ventura (2005, 2008),
   V. Metaftsis, M. Sykiotis (2006, 2008),
   D. Osin and A. Minasyan (2008))

First Theorem

Theorem (*O. Bogopolski, E. Ventura).*
Endomorphisms of torsion-free hyperbolic groups which are pointwise inner on a ball of a uniformly bounded (and computable) radius, are indeed inner automorphisms.

First Theorem

Theorem (*O. Bogopolski, E. Ventura).*
Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Then, there exists a computable constant $\mathcal{C}$ (depending only on δ and the cardinal $\sharp S$) such that, for every endomorphism φ of H, if $\varphi(g)$ is conjugate to g for every element g in the ball of radius $\mathcal{C}$, then φ is an inner automorphism.

Second Theorem

Theorem (*O. Bogopolski, E. Ventura*). *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $a_1, \ldots, a_n$ and $a_{1*}, \ldots, a_{n*}$ be elements of H such that a_{i*} is conjugate to a_i for every $i = 1, \ldots, n$. Then, there is a uniform conjugator for them if and only if $W(a_{1*}, \ldots, a_{n*})$ is conjugate to $W(a_1, \ldots, a_n)$ for every word W in n variables and length up to a computable constant depending only on δ, $\sharp S$ and $\sum_{i=1}^{n} |a_i|$.*

One Corollary

E.K. Grossman proved that if G is a finitely generated conjugacy separable group, then the group $\mathrm{Aut}(G)/\mathrm{Aut}_{\mathrm{pi}}(G)$ is residually finite. From this, one can immediately deduce the following corollary.

Corollary. If H is a torsion-free conjugacy separable hyperbolic group, then $\mathrm{Out}(H)$ is residually finite.

On conjugacy separability for subgroups of virtually free groups

(O.Bogopolski, F. Grunewald)

Notation

Let $H_1, H_2 \leqslant G$.

$H_1 \sim H_2$ if there exists an $g \in G$, such that $g^{-1}H_1g = H_2$.

$H_1 \curvearrowright H_2$ if there exists an $g \in G$, such that $g^{-1}H_1g \leqslant H_2$.

Definition of the CS property for subgroups

G has the *CS-property for finitely generated subgroups*, if for every two finitely generated subgroups H_1, H_2 of G holds:

if $H_1 \not\sim H_2$ in G, then there exists a finite quotient $\overline{G}$ of G, s.t. $\overline{H_1} \not\sim \overline{H_2}$ in $\overline{G}$.

Definition of the ICS-property for subgroups

G has the *ICS-property for finitely generated subgroups*, if for every two finitely generated subgroups H_1, H_2 of G holds:

if $H_1 \not\curvearrowright H_2$ in G, then there exists a finite quotient $\overline{G}$ of G, s.t. $\overline{H_1} \not\curvearrowright \overline{H_2}$ in $\overline{G}$.

Motivation

This paper is motivated by works of

• P.F. Stebe (1972) and J.L. Dyer (1979), who proved that every two elements of a virtually free group can be conjugacy separated,

• F. Grunewald and D. Segal (1978), who proved that every two subgroups of a virtually polycyclic group can be conjugacy separated.

Theorem on the ICS property for subgroups of virtually free groups

Theorem. (*O. Bogopolski, F. Grunewald*)
Let G be a virtually free group, and so it can be represented as the fundamental group of a finite graph of finite groups:
$G = \pi_1(\mathbb{G}, \Gamma, v)$.
If Γ is a tree, G has the (into-)conjugacy property for finitely generated subgroups.

Ingredients

- Coverings of spaces
- Lemma of Serre on fixed points of actions of G on trees
- A theorem on bipartite graphs with long cycles.

Long cycles in bipartite graphs

Erdösh asked.

Füredi, F. Lazebnik, A. Seress, V.A. Ustimenko, A.J. Woldar (1995) answered:

Theorem. For any natural $r, s, t \geqslant 2$, there exists a finite connected bipartite graph of bi-degree r, s, with length of smallest cycle exactly $2t$.

Bild2

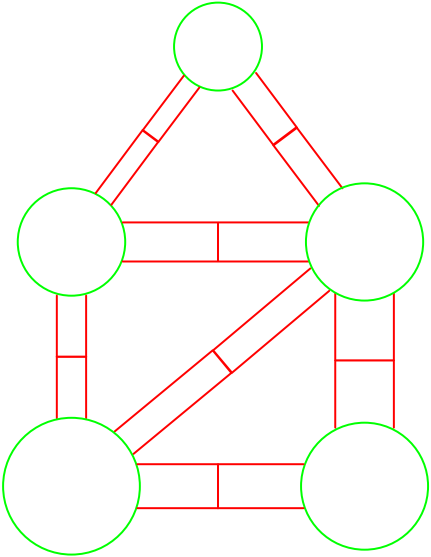
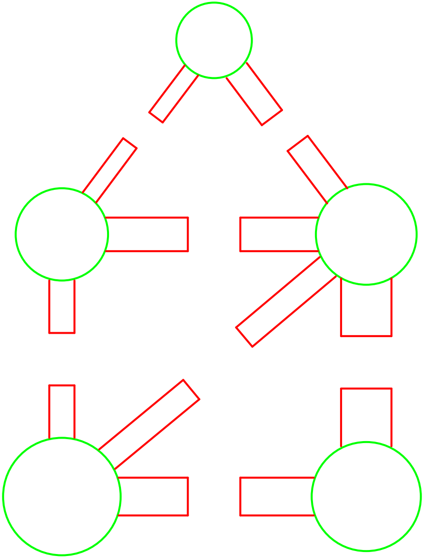


Bild2a



Handles

Let $(\mathcal{B}, b)$ be an elementary piece with only one handel C.

Let $\varphi : (\mathcal{A}, a) \to (\mathcal{B}, b)$ be a covering.

How many and which type of handles contains $\mathcal{A}$?

Handles

Let $(\mathcal{B}, b)$ be an elementary piece with only one handel C.

Let $\varphi : (\mathcal{A}, a) \to (\mathcal{B}, b)$ be a covering.

How many and which type of handles contains $\mathcal{A}$?

Denote

$U = \varphi_*(\pi_1(\mathcal{A}, a))$
$V = \pi_1(\mathcal{B}, b)$.
$W = \pi_1(C, b)$. Clearly $U, W \leqslant V$.

Handles

Let $(\mathcal{B}, b)$ be an elementary piece with only one handel C.

Let $\varphi : (\mathcal{A}, a) \to (\mathcal{B}, b)$ be a covering.

How many and which type of handles contains $\mathcal{A}$?

Denote

$U = \varphi_*(\pi_1(\mathcal{A}, a))$
$V = \pi_1(\mathcal{B}, b)$.
$W = \pi_1(C, b)$. Clearly $U, W \leqslant V$.

Lemma. Handles of $\mathcal{A}$ are in 1-1 correspondence with double classes UgW, $g \in V$. Moreover, $\pi_1(C_{UgW}) \cong g^{-1}Ug \cap W$.

Handles

Let $(\mathcal{B}, b)$ be an elementary piece with only one handel C.

Let $\varphi : (\mathcal{A}, a) \to (\mathcal{B}, b)$ be a covering.

How many and which type of handles contains $\mathcal{A}$?

Denote

$U = \varphi_*(\pi_1(\mathcal{A}, a))$
$V = \pi_1(\mathcal{B}, b)$.
$W = \pi_1(C, b)$. Clearly $U, W \leqslant V$.

Lemma. Handles of $\mathcal{A}$ are in 1-1 correspondence with double classes UgW, $g \in V$. Moreover, $\pi_1(C_{UgW}) \cong g^{-1}Ug \cap W$.

Definition. Two handles C_{Ug_1W} and C_{Ug_2W} have the *same type* if $(g_1^{-1}Ug_1 \cap W) \sim (g_2^{-1}Ug_2 \cap W)$ in W.

Bild1

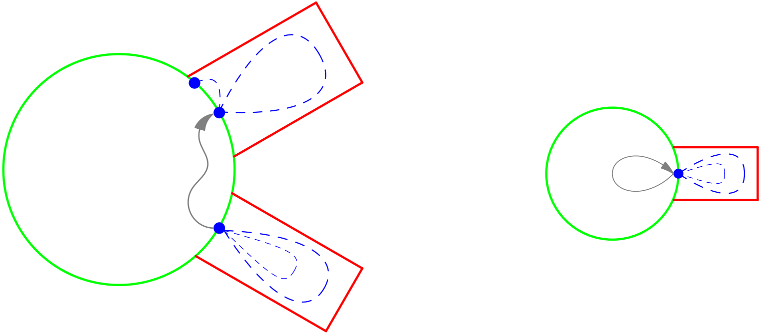


Bild3c

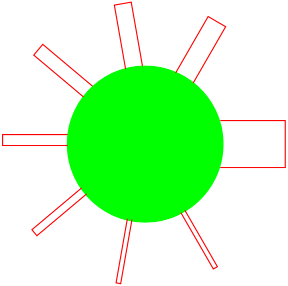


Bild3b

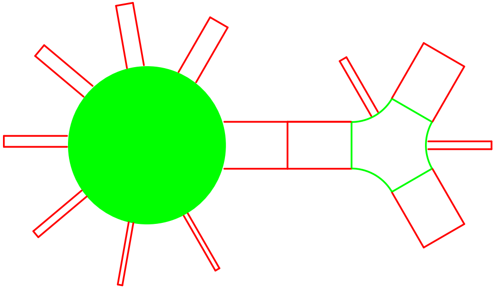


Bild3a

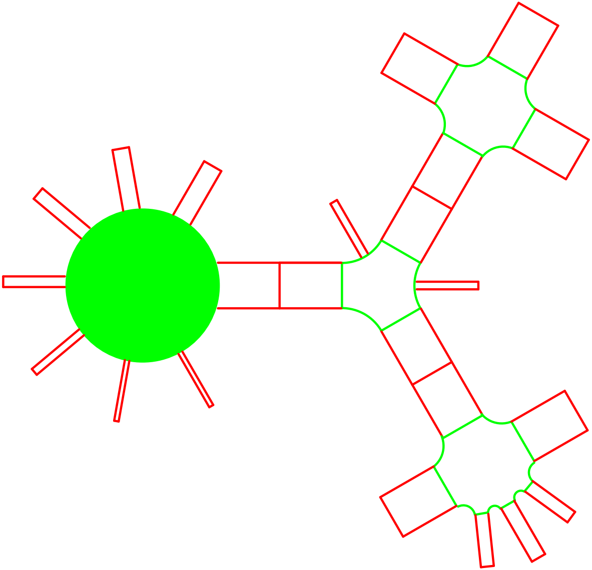


Bild7

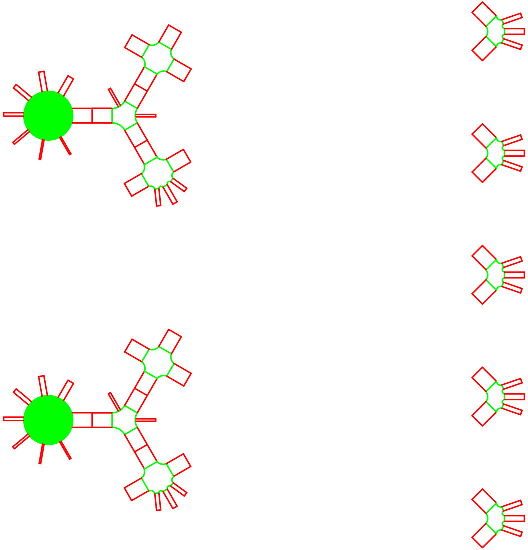


Bild8

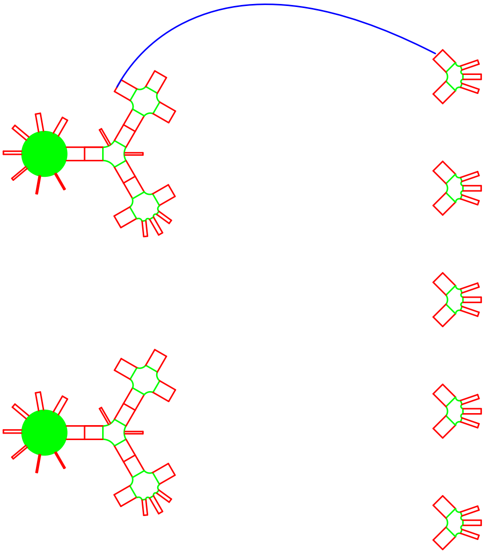


Bild9

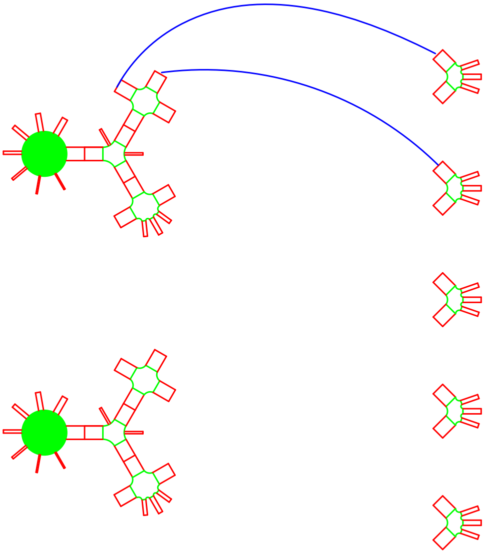


Bild10

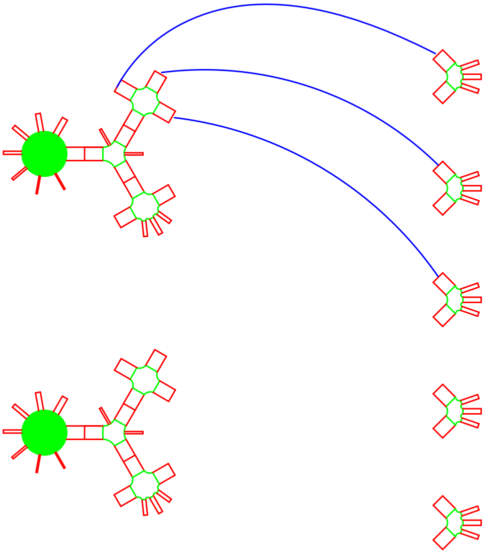


Bild11

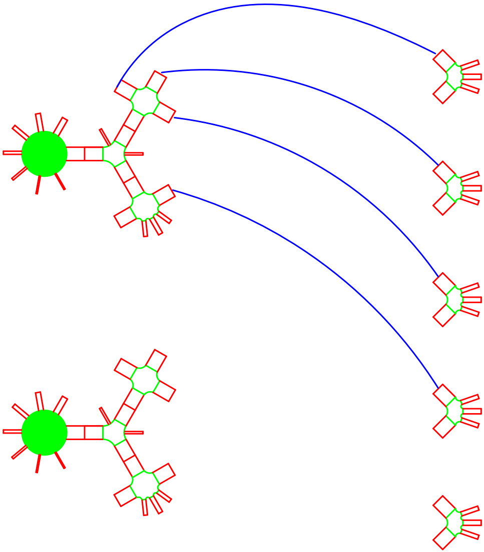


Bild12

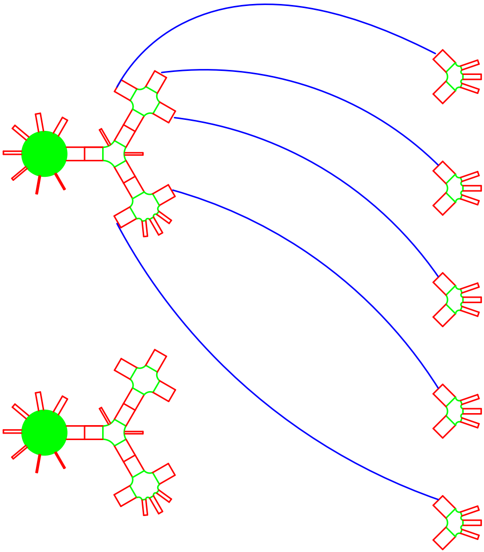
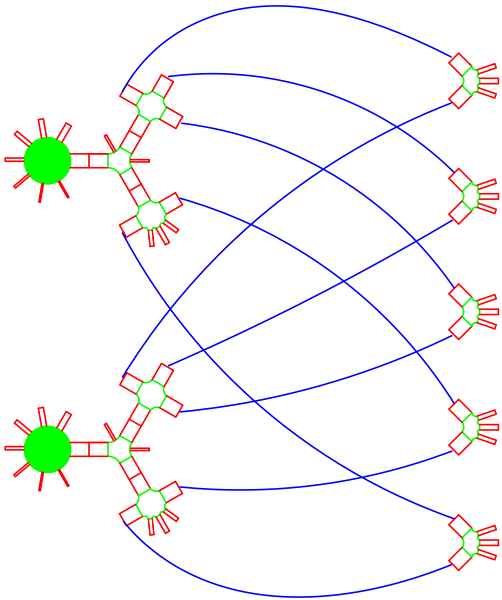


Bild14



Long cycles in bipartite graphs

Erdösh

Füredi, F. Lazebnik, A. Seress, V.A. Ustimenko, A.J. Woldar (1995):

Theorem. For any natural $r, s, t \geqslant 2$, there exists a finite connected bipartite graph of bi-degree r, s, with length of smallest cycle exactly $2t$.

Reformulation a theorem on bipartite graphs

Theorem. For any natural $r, s, t \geqslant 2$, one can glue several r-stars with several s-stars, so that all peripherical vertices of r-stars will be identified with all peripherical vertices of s-stars and the length of smallest cycle in the resulting graph will be exactly $2t$.