Covering $\mathbb{R}$-trees, $\mathbb{R}$-free groups, dendrites, and all that

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[Za] A. Zastrow, *Construction of an infinitely generated group that is not a free product of surface groups and abelian groups, but which acts freely on an $\mathbb{R}$-tree*, Proc. Royal Soc. Edinburg (A), **128** (1998), 433–445.

[BP] V. N. Berestovskii and C. P. Plaut, *Covering $\mathbb{R}$-trees, $\mathbb{R}$-free groups, and dendrites*, preprint arXiv:0904.3767 [math.MG] 23 Apr 2009, 16 p.

We prove that every length space X is the orbit space (with the quotient metric) of an $\mathbb{R}$-tree $\overline{X}$ via a free action of a subgroup $\Gamma(X)$ of isometries of $\overline{X}$.

$\overline{X}$ is defined as the space of based "non-backtracking" rectifiable paths in X, where the distance between two paths is the sum of their lengths from the first bifurcation point to their endpoints.

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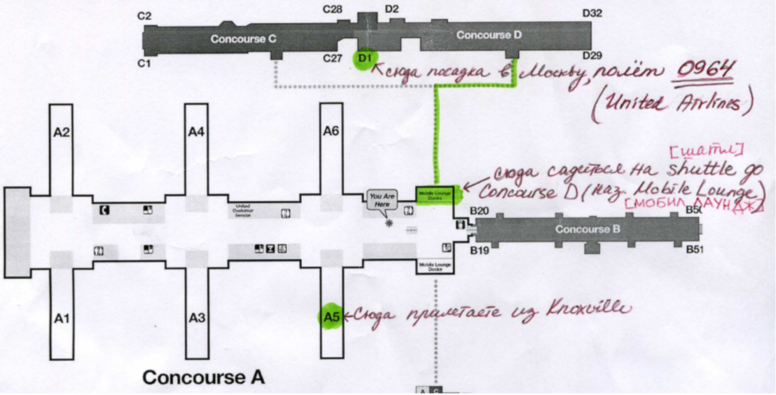

$\Gamma(X) \subset \overline{X}$ is the subset of loops with a natural group structure of "canceled concatenation" and the quotient mapping $\overline{\phi} : \overline{X} \to X$ is the end-point map.

The mapping $\overline{\phi} : \overline{X} \to X$ is a kind of generalized universal covering map called a URL-map, and $\overline{X}$ is the unique (up to isometry) $\mathbb{R}$-tree that admits a URL-map onto X.

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Here a function f between length spaces is called unique rectifiable lifting (URL) if it has the following two properties:

(I) f preserves the length of rectifiable paths in the sense that the length of c is equal to the length of $f \circ c$ for every rectifiable path c in X;

(II) If c is any rectifiable path in Y starting at a point p and $f(q) = p$ then there is a unique path c_L starting at q such that $f \circ c_L = c$, and c_L is rectifiable.

When X is a local $\mathbb{R}$-tree, $\overline{\phi} : \overline{X} \to X$ is the traditional universal covering map and the group $\Gamma(X) = \pi_1(X)$, the fundamental group of X.

When X is a complete Riemannian manifold M^n of dimension $n \geq 2$, or a fractal curve such as the Menger sponge $\mathbb{M}$, the Sierpin'ski carpet S_c or gasket S_g, $\overline{X}$ is isometric to the so-called "universal" $\mathbb{R}$-tree $A_{\mathfrak{c}}$, which has valency the continuum $\mathfrak{c} = 2^{\aleph_0}$ at each point.

Recall that for a point t in a $\mathbb{R}$-tree T, the valency at t is the cardinality of the set of connected components of $T \setminus \{t\}$, and T is said to have valency at most μ if the valency of every point in T is at most μ.

A nontrivial complete metrically homogeneous $\mathbb{R}$-tree can be characterized as a complete $\mathbb{R}$-tree A_μ with valency μ at each point for a cardinal number $\mu \geq 2$.

It is unique up to isometry, and μ-universal in the sense that every $\mathbb{R}$-tree of valency at most μ isometrically embeds in A_μ.

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The existence of A_μ and the results just mentioned were proved in [MNO]; $A_\mathfrak{c}$ can be isometrically embedded at infinity in the Lobachevski space $L^n, n \geq 2$.

For a general separable length space X, $\overline{X}$ is a subtree of $A_\mathfrak{c}$. When X is M^n, $\mathbb{M}$, S_c, or the Hawaiian earring H with a compatible length metric, $\Gamma(X)$ is infinitely generated, locally free, not free, and cannot be presented as a free product of fundamental groups of closed surfaces and abelian groups.

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Moreover, $\overline{X}$ is the minimal invariant $\mathbb{R}$-tree relative to the action of $\Gamma(X)$.

Hence in these cases the action of $\Gamma(X)$ on $\overline{X}$ adds to previous examples of Dunwoody and Zastrow [Za] that give a negative answer to a question of J. W. Morgan.

Indeed, for a particular choice of length metric on H, we obtain precisely Zastrow's example.

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